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OF

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THE

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PURE AND APPLIED MATHEMATICS.

ON THE CONTINUOUS DEFORMATION OF SURFACES.

By DAVID B. MAIR, Christ's College, Cambridge.

THE following is an attempt to shew how surfaces satisfying certain conditions may be deformed by continuous deformation to coincidence with one another. Some properties of surfaces connected with their deformation are added.

1. *Continuous Deformation.* A figure is said to be continuously deformed when it is deformed in such a way that any two points originally contiguous remain contiguous throughout the deformation. Two points on a surface are reckoned *contiguous* when an infinitely short line can be drawn joining them and lying wholly in the surface: thus, if the surface is a plane two points in it are contiguous if the straight line joining them is infinitely short, for this straight line lies wholly in the surface. But two points that occupy the same position in space are not necessarily contiguous. Thus, if two sheets of a surface intersect along a line they are supposed to have no connection along that line, so that a curve which runs to the double line in one sheet and then goes into the other sheet is not a continuous line in the surface: and every point of the double line reckons as two points, one in each sheet, which two points are not contiguous since no line joining them and lying in the surface is infinitely short.

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Hence, on the one hand, no tearing, cutting, or splitting of a surface is allowable: on the other hand double lines are of no account, and of two sheets meeting in such a line either may be deformed as if the other did not exist. For instance, a rectangular strip AC twisted in any manner may by continuous deformation have any number of whole twists removed. The strip in fig. 1 has two whole twists. Let the part at P pass through that at Q, and the strip takes the form of fig. 2. The part AC may then be straightened out by continuous physically possible deformation, and the operation has removed two twists.

Again, in fig. 1 let P pass into Q, but not through, so that there is a double line (fig. 3). Now let the part between P and T be deformed to the left so as to pass bodily through the part between Q and T. The strip has now (fig. 4) no twist between A and D. This part may be straightened out, so that one twist has been removed.

A half twist however, as in fig. 5, cannot be removed by any continuous deformation if the ends A, B are held fixed.

Definition. Two figures are said to be deformed to one another when they are brought to such a position, that with every point of either there coincides one and only one point of the other.

2. *On the Kinds of Surfaces treated.* The surface must be such that if any very small closed curve not meeting itself be drawn cutting out a piece of the surface, the piece cut out is continuously deformable to the piece of a plane bounded by a closed curve not meeting itself. The closed curve on the surface may consist in part of the boundary of the surface, or, in other words, the curve may be an unclosed one having both ends on the boundary.

Let O be any point in the surface. Draw a small curve on the surface, closed if O lie in the interior or having both ends on the boundary if O lie on the boundary, and cutting out a piece containing O. This piece is continuously deformable to a piece of a plane. Let O' be the point in the plane corresponding to O; from O' draw all radii which lie in the piece of the plane, and consider the corresponding lines in the surface. They must all pass through O, and no two can meet at any other point, for then one point on the surface would correspond to two on the plane. To each of these lines draw a tangent at O, ending at O. Since the deformation is continuous and a straight line from O' may be

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supposed to move continuously round O', and trace out that part of the piece of the plane which lies immediately about O', the corresponding line from O on the surface moves continuously and traces out a part of the surface, and the tangent to it moves continuously and traces out a cone whose generators all pass through the vertex O and end there.

For most positions of O this cone is a plane, but at particular points it may be a cone of any kind, provided only that it can be completely described by a straight line from the vertex moving continuously. The *angle of the surface* at the point O is the total angle traced out by this straight line in one complete description of the cone, or is the length of the curve of intersection of the cone by a sphere of unit radius and centre O. When O lies in the interior of the surface the generator returns to its original position and generates a complete cone, and the curve on the unit sphere is closed; but when O lies on the boundary the generator must, to describe the cone completely, begin as tangent to one part of the boundary line and end as tangent to the other part, so that in general the first and last positions are not the same, the cone is not a closed surface and the curve on the sphere not a closed curve.

In general the angle of the surface at a point in the interior is 2π, and at a point in the boundary π. A point in the interior where the angle is not 2π, or on the boundary where the angle is not π, will be called a *Singular Point.*

The form of the surface near the point O is given by the tangent cone. At any point on this cone (except at the vertex) the system of tangent lines forms a plane unless the curve on the unit sphere has a cusp, in which case there are points at which the tangent lines form two half-planes, or unless the cone is not closed, in which case there are points (on the boundary) at which the tangent-lines form a half-plane. In all cases the angle at a point in the interior is 2π, and a point on the boundary π, and infinitely near a singular point, no other singular point can occur, or singular points are isolated points.

Convention for multiple points. A multiple point is one at which a small closed curve does not cut out a part of the above kind, *e.g.* the vertex of a complete quadric cone. A multiple point on a surface is excluded by the condition that a small piece containing it must be deformable to a plane, but with a certain convention such a point may be allowed.

Let the tangent-cone be drawn to the surface at a multiple

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point O, the cone to be considered traced out by generators, infinite in one direction only and ending in the other at O. Let a straight line coinciding originally with one generator trace out as much of the cone as is possible continuously, and let the corresponding part of the surface have no connection at the point O with the rest of the surface. In the same way separate away other parts of the cone till each part so separated may be generated continuously. There finally remains a surface which satisfies the required condition, a surface which has several sheets to all of which a certain point in space belongs. The number of sheets is also given by the number of curves in which the cone is cut by a sphere with centre O. If O lies on the boundary one of these curves will be open; O may lie on the boundary on each of several sheets, then there will be an open curve corresponding to each.

Surfaces having parts at an infinite distance. It is convenient to treat surfaces with all parts at a finite distance. To bring to such form a surface with parts at infinity invert it with respect to a point not on the surface. This brings all parts at infinity to the origin and does not send any other part to infinity: it is also a continuous deformation since it is a conformal deformation. The origin may now be a multiple point, and if so the different sheets must be separated. Or they may be separated before inverting; thus, draw a cone with the origin for vertex through the line of intersection of the surface by the plane at infinity. Separate this cone as before into simple cones and let the part of the surface corresponding to each simple cone be supposed to have no connection at infinity with any other part.

3. *Unifacial and Bifacial Surfaces.* As examples of connected surfaces* may be mentioned (1) a sphere, (2) an anchor-ring, (3) the part of a plane between two concentric circles, *i.e.* a flat ring, (4) the surface formed by cutting a flat ring across, twisting one end through two right angles, and rejoining the ends.

In these examples the last is in a certain way different from the rest, for while they have two distinct faces, which may be called positive and negative faces, on the last it is impossible to make such a distinction. This difference is

* Definitions and propositions assumed may be found in Dr. Forsyth's *Theory of Functions*, Chap. XIV.

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essential and is best defined by Dyck's* *Indicator*. This is a small disc with two radii drawn on it at right angles. Suppose it placed on the surface with the first radius tangential to a closed circuit, and let it move along the surface so that the disc always touches the surface and the first radius always touches the circuit. When the indicator returns to the point of departure the first radius must return to its original position and the second may coincide with its original position or lie in the opposite direction; in the latter case the indicator has been turned over by passage round the circuit, or, since the same face of the disc remains all the time in contact with the surface, the disc has by going round the circuit passed from one side of the surface to the other. If a circuit can be found on a surface, by passage round which the indicator is turned over, the surface is called unifacial; if no such circuit exists the surface is called bifacial.

Points on opposite sides of a surface are not considered separate points, so that two points are *connected* if passage is possible from the first to either side of the surface at the second: for example, the twisted ring, given as last example of connected surfaces, is not *divided* by a cut along the middle line.

(As a "bifacial surface" is more correctly two distinct surfaces and not one connected surface, it might be better to speak of unifacial and bifacial "shells": the former expression is used to avoid the introduction of new terms).

4. *Proposition*:—The $n-1$ cuts to reduce a surface of connectivity n to simple connection may be so drawn that both ends of each cut lie on the original boundary.

Draw on the surface S of connectivity n the line AB of any not-dividing crosscut. If the ends do not both lie on the original boundary the cut must end on itself at B. Let the line instead of ending at B be continued back parallel to itself to near A and then along the original boundary to a point B'. A cut along AB' leaves one connected piece S_1, every boundary line of which consists at least in part of original boundary.

On the surface S_1 make in the same way a cross-cut which has both ends on the boundary of S_1. If the ends do not lie on the original boundary of S, since each boundary line of S_1 consists in part at least of the boundary of S, the ends of the cut may be deformed to lie on the original boundary. This cut being made, there remains a surface S_2, connected, and having

* *Math. Ann.*, vol. 32, p. 457.

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one another but not meeting, the points A_r, B_r, C_r, D_r lying throughout at their proper positions on the curve which deforms to a circle: this is to be done with each of the $n-1$ pairs of curves AB, CD. Throughout the whole deformation the boundary of each of the n pieces remains a closed curve. Since a simply connected surface may be drawn with any closed curve as boundary, each of the simply connected pieces may be continuously deformed so that its boundary coincides all the time with the corresponding one of the above curves, and so that corresponding points of the boundary and the curve always coincide. Two points contiguous on the original surface but separated by a cut remain therefore infinitely near one another, and the deformation may be effected without any of the cuts being made. The remainder of the surface has a circle for boundary and may be made a spherical bowl with this circle for edge. Each of the strips has its breadth changed to its length and becomes a strip which may be supposed, except for twists, to lie on the plane of the circle. The surface has therefore been reduced to the required form by continuous deformation.

Any number of whole twists and knottings of a strip may be removed by passing one part of it through another, and any loopings of strips with one another may be removed by passing them through one another, so that the surface takes the form of fig. 7, which shows part of the edge of the bowl with strips.

(A particular kind of continuous deformation might be considered, namely physically possible deformation: in which no two points of the surface may occupy the same space so that the passage of one part of the surface through another is not allowable. By this deformation any surface, supposed physically possible to begin with, might be reduced to fig. 7, except for twistings, knottings, and loopings of the strips. In this way so simple a surface as an annulus assumes an infinite number of distinct forms according to the number of twistings and knottings, and the general surface seems quite intractable).

6. *Proposition.*—A surface with m boundary lines ($m > 0$) may be deformed to a sphere with m holes, across one of which there are strips so placed that the edges of the strips and of the hole form one boundary line: that is, all boundary lines but one may be reduced to simple holes.

The surface in fig. 7 may be deformed in such a way that the change of *shape* is as if a strip were cut free at one end,

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moved round the boundary-curve to any convenient position and there reunited to the boundary-curve. For, let a simply connected piece containing the end of a strip be cut off, as by cross cuts AB, CD. This piece may be deformed to the form of fig. 8, the points on the edges of cuts being held fixed. Or the deformation may be carried out without making the cut. For shortness this kind of deformation will be spoken of as passing the end of the strip along the boundary.

Now if in fig. 7 there are more than one boundary line, the two edges of some strip belong to different boundary lines. Let $EFGH$ be such a strip, and pass the end EF round in one direction, say EK. Since EKH is a closed curve, the end EF will come to near H. Undo knottings and twistings and loopings, and the strip must become totally without twists as in fig. 9, for if it had a half twist as in fig. 10 the points G, H would lie on the same boundary line, contrary to hypothesis.

Let cuts PQ, RS be made (fig. 9) cutting out a simply connected piece. Keeping the rest of the surface fixed deform this piece to fig. 11, so that points on the cut PQ are unmoved and points on opposite sides of RS originally contiguous remain infinitely near one another. The deformation can be made without the cuts. The hole on which S now lies may be deformed to any convenient position; for draw the line of a cross cut $STUVT$ which begins at S ends on itself and cuts out a simply connected piece containing the hole (fig. 12). It may be deformed to fig. 13, points on $TUVT$ being held fixed.

The surface is now a spherical surface with one simple hole, and another hole across which there are $n-2$ strips and at which there are $m-1$ boundary lines. If $m-1$ is greater than 1, another simple hole may be formed in like manner, and so on until there remains only one boundary line in the region of the strips.

There remains then a spherical surface in which there are m holes; and $m-1$ of these are simple holes, and the last, which may be called the chief hole, has $n-m$ strips across it from edge to edge in such manner that the edges of the hole and strips form one boundary line.

7. *Proposition*:—The part of the surface in the region of the chief hole may be deformed, so that every strip unites diametrically opposite parts of the edge of the hole. And of the strips in this position not more than one is twisted.

A *twisted* strip is to mean a strip which has a half-twist, *i.e.* so twisted that it cannot be straightened out.

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A cut across a strip $ABCD$ may leave the boundary one line or make it two. In the first case, since when the end AB is cut free A, B, C, D lie on the same boundary line, the end AB may be passed round to near C. The strip must then be twisted (fig. 10) as otherwise there would be two boundary lines. In the second case there must be a strip PQ, RS whose two edges belong to the two different boundary lines which the cut makes, and AB, CD lie one on each of these lines. The ends AB, CD may therefore be moved round these two lines to near P and Q (figs. 14, 15). In fig. 14 a cut across $PQRS$ makes two boundary lines, one $PBCQP$ and the other the rest of the boundary: the end RS may therefore be passed round to near AB or CD (figs. 14, 16). In fig. 15 the end PQ may be passed over B and C to near C, in fig. 16 CD may be passed over S and P to between P and B: these give repetitions of fig. 10. Thus, all cases give one of the figures 10 and 14.

In any further deformation of the surface no end of a strip is to be allowed to remain between A and S in fig. 14, or between F and H in fig. 10, but any end passing over A or F must pass on till it gets beyond S or H respectively; this is possible because $ADRQCBPS$ in fig. 14, and $FGEH$ in fig. 10, form continuous lines.

In a similar manner other strips or pairs of strips may be separated from the rest, till there only remain (1) pairs of untwisted strips as in fig. 14, and (2) single twisted strips as in fig. 10. The four ends of a pair of strips may without altering the form of the pair be passed over any number of single strips, and so all pairs be brought together and also all single strips (fig. 17).

Let there be r single strips A_1B_1, A_2B_2, ..., A_rB_r, and s pairs of strips C_1E_1 and D_1F_1, C_2E_2 and D_2F_2, ..., C_sE_s and D_sF_s (fig. 17). Pass the end B_1 over A_2 and B_2 to lie between A_2 and B_2, pass the end B_2 over A_3 and B_3 to lie between A_3 and B_3, and so on, last of all B_{r-1} passing over A_r and B_r. Each strip is untwisted as it passes over the next, so that the only strip remaining twisted is A_rB_r (fig. 18). Next pass A_3 over $B_2A_2B_1A_1A_2B_2$ to lie between B_3 and A_4. This separates off a pair of strips A_1B_1, A_2B_2. In this way separate off pairs of strips till only A_rB_r or A_rB_r and $A_{r-1}B_{r-1}$ are left. There then are a number of pairs of strips, and either one twisted strip or a twisted strip paired with a straight one.

The case may arise in which fig. 17 contains no twisted strip. Hence, every surface is deformable to a spherical surface, having a certain number of simple holes and one chief

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hole, which has its edges joined by (1) a number of pairs of untwisted strips, and (2) no other strip, or one twisted strip, or a pair of strips one of them twisted.

The surface in the first case is obviously bifacial, one face consisting of the outside of the spherical surface and one side of each strip, the other of the inside of the spherical surface and the other side of each strip; the surfaces in the other cases are unifacial, for a circuit over the twisted strip turns the Indicator. If the connectivity is n and the number of boundary lines m, the number of strips has been shewn to be $n-m$; and any bifacial surface is deformable to a surface that has an even number of strips, so that for a bifacial surface $n-m$ is even. Also a unifacial surface is deformable to a surface that has a twisted strip, so that for a unifacial surface $n-m$ is not less than 1.

To continue the deformation let the pairs of untwisted strip be called $\alpha_1\beta_1$, $\alpha_2\beta_2$, ..., $\alpha_{2t}\beta_{2t}$, the single twisted strip $\alpha_{2t+1}\beta_{2t+1}$, and the pair one of which is twisted $\alpha_{2t+1}\beta_{2t+1}$ and $\alpha_{2t+2}\beta_{2t+2}$; so that the order of the ends round the chief hole is

$$\alpha_1\alpha_2\beta_1\beta_2\alpha_3\alpha_4\beta_3\beta_4\ldots\alpha_{2t}\beta_{2t}\alpha_{2t+1}\alpha_{2t+2}\beta_{2t+1}\beta_{2t+2}.$$

Pass the ends α_3 and α_4 over $\beta_2\alpha_3\alpha_1\beta_1$ to lie between α_2 and β_1. Next pass the ends α_5 and α_6 over $\beta_4\alpha_4\alpha_3\beta_3$, and over $\beta_2\alpha_3\alpha_1\beta_1$ to lie between α_4 and β_1. Next pass α_7 and α_8 over $\beta_6\alpha_6\alpha_5\beta_5$, $\beta_4\alpha_4\alpha_3\beta_3$, $\beta_2\alpha_3\alpha_1\beta_1$ to lie between α_6 and β_1, and so on. The last step is in the three cases

(1) α_{2t-1} and α_{2t} go over $\beta_{2t-2}\alpha_{2t-2}\alpha_{2t-3}\beta_{2t-3}\ldots\beta_2\alpha_2\alpha_1\beta_1$,

(2) α_{2t+1} goes over $\beta_{2t}\alpha_{2t}\alpha_{2t-1}\beta_{2t-1}\ldots\beta_2\alpha_2\alpha_1\beta_1$, and

(3) α_{2t+1} and α_{2t+2} go over $\beta_{2t}\alpha_{2t}\alpha_{2t-1}\beta_{2t-1}\ldots\beta_2\alpha_2\alpha_1\beta_1$.

The ends then lie round the circle in the required order:

(1) $\alpha_1\alpha_2\ldots\alpha_{2t}\beta_1\beta_2\ldots\beta_{2t}\alpha_1$,

(2) $\alpha_1\alpha_2\ldots\alpha_{2t}\alpha_{2t+1}\beta_1\beta_2\ldots\beta_{2t}\beta_{2t+1}\alpha_1$, or

(3) $\alpha_1\alpha_2\ldots\alpha_{2t}\alpha_{2t+1}\alpha_{2t+2}\beta_1\beta_2\ldots\beta_{2t}\beta_{2t+1}\beta_{2t+2}\alpha_1$,

and by moving round the edge of the hole they come to the required position. The surface is then shewn by fig. 19, where the strip not completely drawn may be straight or twisted. When it is straight the number of strips is even, when twisted the number may be odd or even.

It has been shewn that any bounded surface is continuously

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deformable to a spherical surface which has $m-1$ simple holes, and one other hole across which there lie $n-m$ strips joining diametrically opposite parts of the edge of the hole, and being either (1) all straight and even in number, or (2) all but one straight and odd or even in number: the former or the latter being the form of the surface according as it is bifacial or unifacial ; n being the connectivity of the surface and m the number of its boundary lines.

These will be called the standard forms for bounded surfaces.

Closed Surfaces. Take the standard form of surface, bifacial or unifacial, of connectivity n, and number of boundaries 1. With a point in the axis of the chief hole as vertex and the boundary line as base describe a cone. Let this simply connected cone be added to the surface. There results a closed surface, bifacial or unifacial, of connectivity n. This will be taken as the standard form for closed surfaces.

To deform a closed surface to the standard form, cut out a simply connected piece by a loop cut. The remainder of the surface is of connectivity n and has one boundary; deform it to its standard form. The simply connected piece may be deformed to the conical piece of the standard form for closed surfaces, points of the original surface originally contiguous remaining infinitely near together; so that the whole deformation may be carried out without any cut being made.

8. *Proposition*:—The necessary and sufficient conditions that one surface may be continuously deformed to another are

(1) they must be of the same kind, both bifacial or both unifacial;

(2) they must have the same connectivity;

(3) they must have the same number of boundaries.

If the first surface is deformed to the second, any circuit which turns the indicator on one of them does so on the other, and therefore they are both bifacial or both unifacial; if any system of cross cuts reduce one of the surfaces to one simply connected piece, they reduce the other to one simply connected piece, therefore the number of crosscuts to reduce to simple connection are equal, that is, their connectivities are equal; and each boundary line of one surface coincides with one and only one of the other surface, therefore the number of boundary lines are equal. The conditions are therefore necessary.

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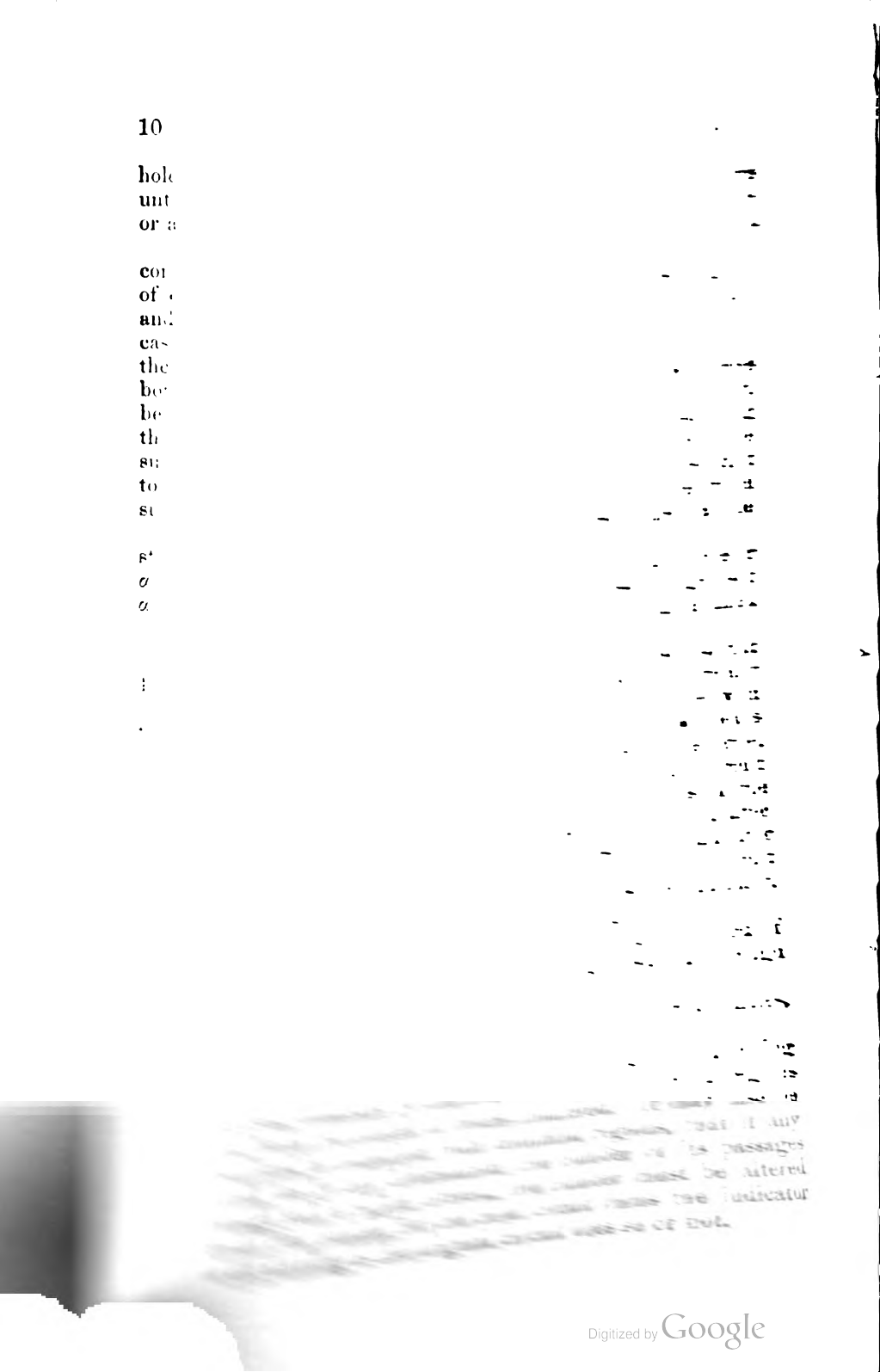

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Again, a cut from a hole through a handle, ending on itself, diminishes the number of handles by one, and increases the number of holes by one. Thus β cuts of the first kind and α of the second leave a spherical surface with $\alpha+\gamma$ simple holes: $\alpha+\gamma-1$ cuts joining these holes make the surface simply connected, and the original surface has therefore connectivity $2\alpha+\beta+\gamma$. And this surface is bifacial only if $\beta=0$.

Hence, a bifacial surface of connectivity n and having m boundary lines may be represented as a sphere with $\frac{1}{2}(n-m)$ handles (for $n-m$ is even) and m simple holes.

The surface again represents a unifacial surface of connectivity n and number of boundary lines m if $\gamma=m$ and $2\alpha+\beta=n-m$. In particular (1) a sphere with $n-m$ unifacial regions and m simple holes, and (2) a sphere with one unifacial region, $\frac{1}{2}(n-m-1)$ handles and m holes, or two unifacial regions, $\frac{1}{2}(n-m-2)$ handles and m holes, according as $n-m$ is odd or even, both represent the unifacial surface with connectivity n and number of boundary lines m.

Hence, the conditions that for a bifacial surface $n-m$ is zero or an even positive integer, and for a unifacial surface $n-m$ is not less than 1, are the only conditions on m and n. For, data being given subject to these conditions, the surface may be constructed as above.

Properties of Surfaces.

10. A crosscut increases by 1, diminishes by 1, or leaves unaltered the number of boundary lines; a loopcut increases the number by either 1 or 2.

First, let a crosscut AB have its ends on two different boundary lines PQ, RS, and let the two edges of the cut be called A_1B_1 and A_2B_2. Then a point passing round the boundary line to which A_1B_1 belongs will after leaving A_1B_1 describe the line RS to B_2, return along B_2A_2 and A_2QPA_1. The two boundary lines and the edges of the cut from one boundary line, so that the number of boundary lines is reduced by unity.

Next let the cut have both ends on the boundary line PQ (fig. 21). The circuit $ACBQA$ may or may not turn the indicator.

In the latter case the indicator setting out from A_2 along the boundary A_2Q and having its second radius in the surface will upon arriving at A coincide with its former position and still lie in the surface; it must therefore arrive at A_2 and not A_1, and A_1 and A_2 belong to different boundary

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lines. In the former case the second radius does not after one circuit return to its former position, but still lies in the surface; so that the indicator must arrive at A_1. It will then describe A_1PBCA_2, so that the whole forms one boundary line.

Lastly consider a loopcut. In the same way its two edges will form one or two boundary lines according as a circuit round it turns the indicator or not.

There being no circuit with turning indicator on a bifacial surface, the number of boundary lines on it will be either increased or diminished by unity by a crosscut, and increased by two by a loopcut.

11. A connected surface cannot be divided by (1) a crosscut with ends on different boundaries, or (2) a crosscut such that the circuit composed of the line of the crosscut and either part of the boundary turns the indicator, or (3) a loopcut the circuit of which turns the indicator.

For in each case the opposite edges of the cut belong to the same boundary line, therefore the parts adjacent to the edges are connected and the surface is not divided.

12. On a bifacial surface $n-m$ is zero or an even positive integer, on a unifacial surface $n-m$ is a positive integer; n being the connectivity of the surface and m the number of boundary lines.

On the bifacial surface $m-1$ crosscuts joining the m boundary lines may be made without dividing the surface, so that n is not less than m. Any crosscut on the surface either increases or diminishes the number of boundary lines by unity. Therefore if $n-1$ crosscuts be made the number of boundary lines added or subtracted differs from $n-1$ by an even number. And if these cuts be made so as to reduce the surface to simple connection there remains one boundary line; therefore

$$m \pm (n - 1 \pm 2k') = 1,$$

k' being an integer: or since $n-m$ is not negative,

$$n - m = 2k,$$

where k is zero or a positive integer.

On the unifacial surface there is a circuit which turns the indicator. A loopcut along this circuit does not divide the surface, does not affect the connectivity, and increases the number of boundary lines by unity. Make a crosscut from

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this loopcut to the original boundary, and the two cuts being reckoned as one, one crosscut has been made leaving a surface of m boundary lines. Make $m-1$ crosscuts joining these boundary lines. Then m crosscuts have been made without dividing the surface, and n is not less than $m+1$.

13. When all crosscuts have been made in succession that are possible without dividing the surface, the surface is simply connected. When all loopcuts are made in succession that are possible without dividing the surface, the surface is deformable to a spherical surface with a number of simple holes. The number of crosscuts is constant and equal to $n-1$, the number of loopcuts on a bifacial surface is constant equal to $\frac{1}{2}(n-m)$, and on a unifacial surface may be any integer between $\frac{1}{2}(n-m)$ and $n-m$ inclusive; where n is the connectivity, and m the number of boundary lines.

The statement concerning crosscuts, so far as it is not a definition, has already been proved. Consider the crosscuts annulled in succession, and we have Dyck's method, who classifies surfaces according to the number of *joinings* necessary to build them up from simply connected surfaces.

Suppose all possible not-dividing loopcuts have been made. There being no further not-dividing loopcut the surface is bifacial. Let there be m' boundaries. Make $m'-1$ crosscuts joining the boundary lines. No not-dividing loopcut can now be drawn, since the same loopcut was before impossible, and therefore no not-dividing crosscut can be drawn with its ends on the same boundary line. There is only one boundary line and therefore the surface is simply connected. Hence, before the crosscuts were made, it had connectivity m', had m' boundary lines, and was bifacial. A sphere with m' simple holes has these three properties, and the surfaces are equivalent.

Again, let s of the loopcuts be along circuits with turning indicator and t along circuits with not-turning indicator. These cuts then add $s+2t$ to the number of boundaries, and $m+s+2t-1$ crosscuts reduce the surface to simple connection. Therefore

$$m+s+2t=n,$$

and the number of loopcuts

$$=s+t$$

$$=n-m-t=\tfrac{1}{2}(n-m+s).$$

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direction by $B_2A_2 \ldots B_r$, one end of each other cut, and in the other direction by $A_2B_2 \ldots A_r$, the other end of each cut, and any two points A_t, B_t, ends of the same cut, are separated in each direction by one end of each of the other cuts.

If the surface is unifacial the order of the points round the boundary is

$$A_1B_2A_2 \ldots B_{r-2}A_{r-1}A_rB_{r-1}A_{r-2} \ldots B_2A_2B_1B_rC_1C_2 \ldots C_rA_1,$$

or $$A_1B_2A_2 \ldots A_{r-2}B_{r-1}A_rA_{r-1}B_{r-2} \ldots B_2A_2B_1B_rC_1C_2 \ldots C_rA_1,$$

according as r is even or odd. Thus $C_1C_2 \ldots C_r$ lie together; the points A_1, B_1 are separated on one part by B_r only, the points A_2, B_2 by $B_rA_1B_1$, the points A_t, B_t (t not $=r$) by $B_rA_1B_1A_2B_2 \ldots A_{t-1}B_{t-1}$, and A_r, B_r by one end of each of the other cuts.

If the chief hole be deformed to a circle, the order of the points is given by the following: for the bifacial surface A_t and B_t ($t = 1, \ldots r$) are at the ends of a diameter; for the unifacial surface A_r and B_r are at the ends of a diameter, and A_tB_t ($t = 1, \ldots r-1$) at the ends of an ordinate to that diameter.

15. A cross-cut which makes a unifacial surface bifacial has both ends on the same boundary line. The circuit composed of the crosscut and a part of the boundary line turns the indicator or not according as $n - m$ is odd or even. And a unifacial surface may by one cut be made bifacial, or by $n - 2$ crosscuts be made doubly connected and remain unifacial.

First consider a crosscut with ends on different boundary lines. If there is a circuit with turning indicator whose path is not crossed by this cut the surface remains unifacial. If a circuit with turning indicator is crossed by the cut as at P, S and so prevented, from P draw a line $PQRS$ along the cut to the boundary line at Q, round the boundary line to R, and back along the cut to S. This line completes a circuit; and this new circuit turns the indicator, for the old one did, the parts PQ, RS counteract one another, and the remaining part QR is along a boundary line and so cannot turn the indicator. Therefore such cut cannot make the surface bifacial.

Again, let a crosscut make the surface bifacial. It reduces the connectivity by 1, and leaves the number of boundaries unaltered or increases it by 1. Hence, since on a bifacial surface the connectivity differs from the number of boundary-lines by an even integer, if $n - m$ is odd, the number of

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boundary lines is unaltered, and the circuit turns the indicator; if $n-m$ is even, the number of boundary lines is increased by 1, and the circuit does not turn the indicator.

Also a surface can be made bifacial by one cut, for the crosscut in the standard set across the twisted strip makes the surface bifacial. Deform this crosscut to end on itself and annul a part of it to leave a loopcut, and the loopcut left makes the surface bifacial. If on the other hand all the crosscuts are of the standard form, except that across the twisted strip, there remains a unifacial doubly connected surface.

16. A cut which makes a unifacial surface bifacial crosses any circuit with turning indicator an odd number of times. A dividing cut crosses every circuit an even number of times. Hence the same cut cannot make a surface bifacial and divide it.

Let a cut making bifacial meet a circuit with turning indicator in points AA', BB'... these points being in order along the circuit. Let AB and also $A'B'$ be joined by lines along the cut. There are then two circuits, of which one, $A'B'C'$, is not prevented by the cut and therefore does not turn the indicator. The other, being formed from the original circuit by removing a circuit which does not turn the indicator, must turn the indicator. In this way remove circuit after circuit; the finally remaining circuit turns the indicator and has either no crossing or one crossing with the cut. The former is impossible, for there would then be a circuit with turning indicator on a bifacial surface. The number of crossings is therefore odd.

Consider a dividing cut and any circuit. Let the circuit be described from any point lying in the first piece of the divided surface. The circuit ends in this piece, and whenever it crosses the cut it enters the other piece and must recross to return to the first piece; and the number of crossings is even.

Hence a dividing cut cannot make a surface bifacial, or if a cut divide a unifacial surface one part remains unifacial.

17. When a cut makes a unifacial surface bifacial, the parts of the surface adjoining the two edges of the cut belong to opposite faces of the bifacial surface.

As in the above a circuit can be found with turning indicator and crossing the cut once only, as at PQ. After the cut is made, P and Q are connected by this circuit. And since it turns the indicator the upper side at P and the lower at Q belong to the same face of the bifacial surface; or, more accurately, the normals at P and Q to the positive face of the bifacial surface lie in opposite directions.

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18. The necessary and sufficient conditions that a set of cuts which reduce a surface to simple connection are a standard set, are:—

(1) All cuts have one end at least on a certain boundary-line, and no other boundary-line has more than one end of one cut on it.

(2) For a bifacial surface the cuts which have both ends on the chief boundary line have their ends so arranged that on either route along the boundary from one end of the cut to the other there lies one end of each of the other cuts.

(3) For a unifacial surface one of the cuts by itself makes the surface bifacial and on either route from one of its ends to the other there lies one end of each of the other cuts, and on a route between the ends of any other cut, there must lie one end only of the cut which makes the surface bifacial, and both or neither of the ends of any third cut.

(4) The ends of the cuts which have one end only on the chief boundary line lie all together, not separated by the end of any other cut; and on a unifacial surface one end of the cut which makes bifacial lies next them.

First, these conditions are necessary, for they are satisfied by the standard set on the standard form of surface.

Next, for the bifacial surface they are sufficient. On such surface call the ends of the cuts which have one end only on the chief boundary line $C_1, C_2, \ldots, C_r$, in the positive direction along the line, and let the other ends of these cuts be called $D_1, D_2, \ldots, D_r$ respectively. Proceeding from C_r name the other ends $A_1, B_1, A_2, \ldots, A_{r-1}, B_r, B_1, A_2, B_2, \ldots, B_{r-1}, A_r$, in order: then A_i and B_i are ends of the same cut. Let all the cuts be made. The direction at every point of the original boundary which was the positive direction is still the positive direction, so that a positive description of the boundary passes over the points in the order

$$C_r A_1 B_1 A_2 B_2 A_3 B_3 \ldots A_r B_r B_1 A_1 B_2 A_2 \ldots$$

$$B_r A_r C_1 D_1 D_1 C_1 C_2 D_2 D_2 C_2 \ldots C_r D_r D_r C_r.$$

The points occur in the same order as when the standard surface is reduced by the standard set of cuts to simple connection. Therefore by deforming to a spherical bowl and reuniting the cuts, the surface becomes the standard surface with the cuts in the standard form.

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Lastly, consider the unifacial surface. On it there is no distinction of positive and negative directions along a boundary-line. Denote by B_r the end of the cut which makes bifacial, which lies next the ends of cuts which have one end only on the chief boundary line, and name these ends $C_1, C_2, \ldots, C_s$ in order from B_r. For convenience call $B_r C_1 C_2 \ldots C_s$ the positive direction, and let the other ends of these cuts be called $A_r, D_1, D_2, \ldots, D_s$ respectively. Call the ends after C_s in order $A_1, B_2, A_3, B_4, \ldots$, and in the negative direction from B_r call them $B_1, A_2, B_3, \ldots$. Then, if the conditions are satisfied A_t and B_t are ends of the same cut.

Let the cut $A_r B_r$ be made. It makes one boundary-line

$$A_1 B_2 A_3 \ldots A_r B_r B_1 B_2 B_3 \ldots A_r B_r C_1 C_2 \ldots C_s,$$

or two

$$A_1 B_2 A_3 \ldots A_r B_r C_1 C_2 \ldots C_s \text{ and } B_1 A_2 B_3 \ldots A_r B_r B_1,$$

according as r is odd or even. And if $C_1 \ldots C_s$ is chosen as the positive direction on the bifacial surface, the positive direction is A_r to B_r on each edge of the cut, and the boundary-lines above are named in their positive directions.

In the former case the ends of the cuts not yet made lie on the boundary-line in the order

$$A_1 B_2 A_3 \ldots A_{r-2} B_{r-1} B_1 A_2 B_3 \ldots B_{r-2} A_{r-1} C_1 \ldots C_s A_1$$

and the other ends $D_1, \ldots, D_s$ one on each of the other boundary-lines. This is a standard set of cuts for a bifacial surface. Deform it then to the standard form (fig. 22). The two edges of the cut $A_r B_r$ lie between A_{r-1} and C_1, and between B_{r-1} and B_1 respectively; and by rejoining them the surface is restored in the standard form with the given cuts as standard set.

In the second case after the cut $A_r B_r$ is made the points A_{r-1}, B_{r-1} lie on different boundary-lines. Make the cut $A_{r-1} B_{r-1}$ and the two become one boundary-line

$$A_1 B_2 A_3 \ldots B_{r-2} A_{r-1} B_{r-1} A_r B_r B_1 \ldots A_{r-2} B_{r-1} A_{r-1} A_r B_r C_1 \ldots C_s A_1.$$

The ends of cuts not yet made lie along the boundary-line in the order

$$A_1 B_2 A_3 \ldots A_{r-3} B_{r-2} B_1 A_2 B_3 \ldots B_{r-3} A_{r-2} C_1 \ldots C_s A_1,$$

that is, in the standard position. Deform this bifacial surface to the standard form with the crosscuts in the standard position. The edges of the cuts $A_{r-1} B_{r-1}$ and $A_r B_r$ lie between

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A_{r-2} and C_1 in the order $B_{r-1}A_{r-1}A_rB_r$, and between B_{r-1} and B_1 in the order $A_{r-1}B_{r-1}A_rB_r$. Rejoin these edges, the strip cut by $A_{r-1}B_{r-1}$ is straight and the other twisted. And the surface is restored in the standard form, the crosscuts also being in the standard form.

19. On a bifacial surface any not-dividing crosscut may be taken as one of a standard set. On a unifacial surface any crosscut which makes the surface bifacial, any crosscut with ends on the same boundary if the corresponding circuit turns the indicator, and any crosscut with ends on different boundaries, may be taken as one of a standard set. The crosscut is of course supposed to have both ends on the original boundary.

On a bifacial surface let a crosscut C_1D_1 have ends on different boundaries. Let there be $s+1$ boundary lines and connectivity $r+s+1$. In the positive direction of the boundary line from the point C_1, take points

$$C_2C_3\ldots C_sA_1B_2A_3\ldots A_{r-1}B_rB_1A_2\ldots A_r$$

in order. D_1 lies on one of the other boundary lines, on the remaining boundary lines take points $D_2\ldots D_s$. Make cuts C_2D_2, C_3D_3, ..., C_sD_s; they do not divide the surface, but reduce the number of boundaries to unity and the connectivity to $r+1$. This being greater than 1, a crosscut with both ends on the boundary line can be made without dividing the surface, and its ends deformed to A_1 and B_1. Then there are two boundary lines, $B_2A_2B_2\ldots B_r$ and $A_2B_2\ldots A_r$. Make a cut A_2B_2 and there results a surface of connectivity $r-1$ and one boundary line. In this way $r+s-1$ other crosscuts may be drawn to form with C_1D_1 a standard set.

Let there be a crosscut A_1B_1 on a bifacial surface, not dividing it, and having both ends on the same boundary-line. Take points A_2, ..., A_r, B_2, ..., B_r, C_1, ..., C_s on this line, so that the order is

$$A_1B_2A_3\ldots A_{r-1}B_rB_1A_2\ldots A_rC_1\ldots C_s;$$

and on the other s boundary lines points $D_1\ldots D_s$ one on each. Since A_1B_1 does not divide the surface, cuts C_1D_1, ..., C_sD_s may be made, not crossing the line A_1B_1. Then A_1B_1 may be made and any cuts A_2B_2, ..., A_rB_r as before. And a standard set has been drawn of which A_1B_1 is one cut.

On a unifacial surface with $s+1$ boundary lines and connectivity $r+s+1$, let a cut A_rB_r be given which makes it

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bifacial. On the boundary-line on which A_r and B_r lie, take points so that the order is in one direction,

$$B_r C_1 C_2 \dots C_s A_1 B_2 A_3 \dots A_r,$$

and in the other

$$B_r B_1 A_2 B_3 \dots A_r;$$

and on the other s boundary-lines points $D_1 \dots D_s$. If the cut $A_r B_r$ leaves the boundary-line one, the other cuts $A_1 B_1, \dots, A_{r-1} B_{r-1}, C_1 D_1, \dots, C_s D_s$ may then be made as above. If the cut $A_r B_r$ makes the boundary-line two, A_{r-1} and B_{r-1} lie one on each of these and a cut $A_{r-1} B_{r-1}$ may be made. This cut joins the two boundary-lines again into one; and the other cuts $C_1 D_1, \dots, C_s D_s, A_1 B_1, \dots, A_{r-2} B_{r-2}$ may then be made as above.

If the given cut $C_1 D_1$ join two boundaries, take on the first points so that

$$C_1 C_2 \dots C_s A_1 B_2 A_3 B_4 \dots A_r \dots A_4 B_3 A_2 B_1 B_r$$

are in order; and on the remaining $s-1$ boundary-lines points $D_2 D_3 \dots D_s$. Make the cut $C_1 D_1$ and then a cut $A_r B_r$ making the surface bifacial. $A_r B_r$ drawn alone would make the surface bifacial, since a cut $C_1 D_1$ joining two boundary lines cannot. Therefore $A_r B_r$ may be considered drawn first, and the standard set completed as before.

If a cut $A_1 B_1$ be given, with ends on the same boundary, not making the surface bifacial, and the corresponding circuit turns the indicator, take points on the same boundary line, so that

$$A_1 B_2 A_3 B_4 \dots A_r \dots A_4 B_3 A_2 B_1 B_r C_1 C_2 \dots C_s$$

are in order. Make first the cut $A_1 B_1$ and then $A_r B_r$, reducing the surface to be bifacial, (this being possible since the cut $A_1 B_1$ leaves A_r and B_r on the same boundary-line). Suppose the surface near this part coincides with the plane of the paper, and take $B_r A_1$ (fig. 23) as the positive direction round the boundary of the reduced surface. Then since the circuit corresponding to the cut $A_1 B_1$ turns the indicator, the positive direction is $B_r A_1 B_1 A_r$. Therefore the positive face at the part Q is downwards. And since the positive face faces different ways on opposite sides of the cut $A_r B_r$, the positive face at S is downwards, and at R upwards. Hence, if the cut $A_1 B_1$ is rejoined it joins positive face with positive face and does not make the surface unifacial. Therefore the cut $A_r B_r$ by itself reduces the surface to be bifacial. It may be

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supposed made first and the complete set of cuts drawn as before, A_1B_1 being one of them.

The cut A_1B_1 could not be one of a standard set if the corresponeing circuit did not turn the indicator. For it would then make two boundary-lines, A_r and B_r would lie one on each of these, and no cut A_rB_r could make the surface bifacial.

20. *Irreducible Circuits.* For a surface with $s+1$ boundaries and connectivity $r+s+1$ the following will be taken as standard system of irreducible circuits:

First, r circuits each along one of the r strips in the standard figure and being a great circle on the sphere, and

Second, s circuits each being a circle on the sphere of small radius and enclosing one of the simple holes as in fig. 19.

21. On a bounded unifacial surface a simple complete irreducible system may consist of circuits of which all, or only one, or any immediate number, turn the indicator.

A complete irreducible system is *simple* when any circuit may, without being repeated, be resolved into a combination of the circuits of the system.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be any such system. The system $\alpha_1, \alpha_2+\lambda_2\alpha_1, \alpha_3+\lambda_3\alpha_1, \ldots, \alpha_{n-1}+\lambda_{n-1}\alpha_1$ is also simple, for any circuit equivalent to a circuit composed of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ is also equivalent to a circuit composed of $\alpha_1, \alpha_2+\lambda_2\alpha_1, \ldots, \alpha_{n-1}+\lambda_{n-1}\alpha_1$.

Let $\alpha_1, \ldots, \alpha_{n-1}$ be the standard system, α_{n-1} that circuit along the twisted strip. Then α_{n-1} alone turns the indicator. Add it to any of the other circuits and the new system is simple: so that the number with turning indicator can be made any number between one and all inclusive.

The circuits cannot all have not-turning indicator, for any circuit composed of them could not have turning indicator while such circuits exist on the surface. The system $\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, 2\alpha_{n-1}$, has no circuit with turning indicator, and is complete and irreducible, but not simple.

22. On a closed bifacial surface, the assignment of a point-boundary does not alter the number of circuits in a complete irreducible system. On a closed unifacial surface it increases the number by unity.

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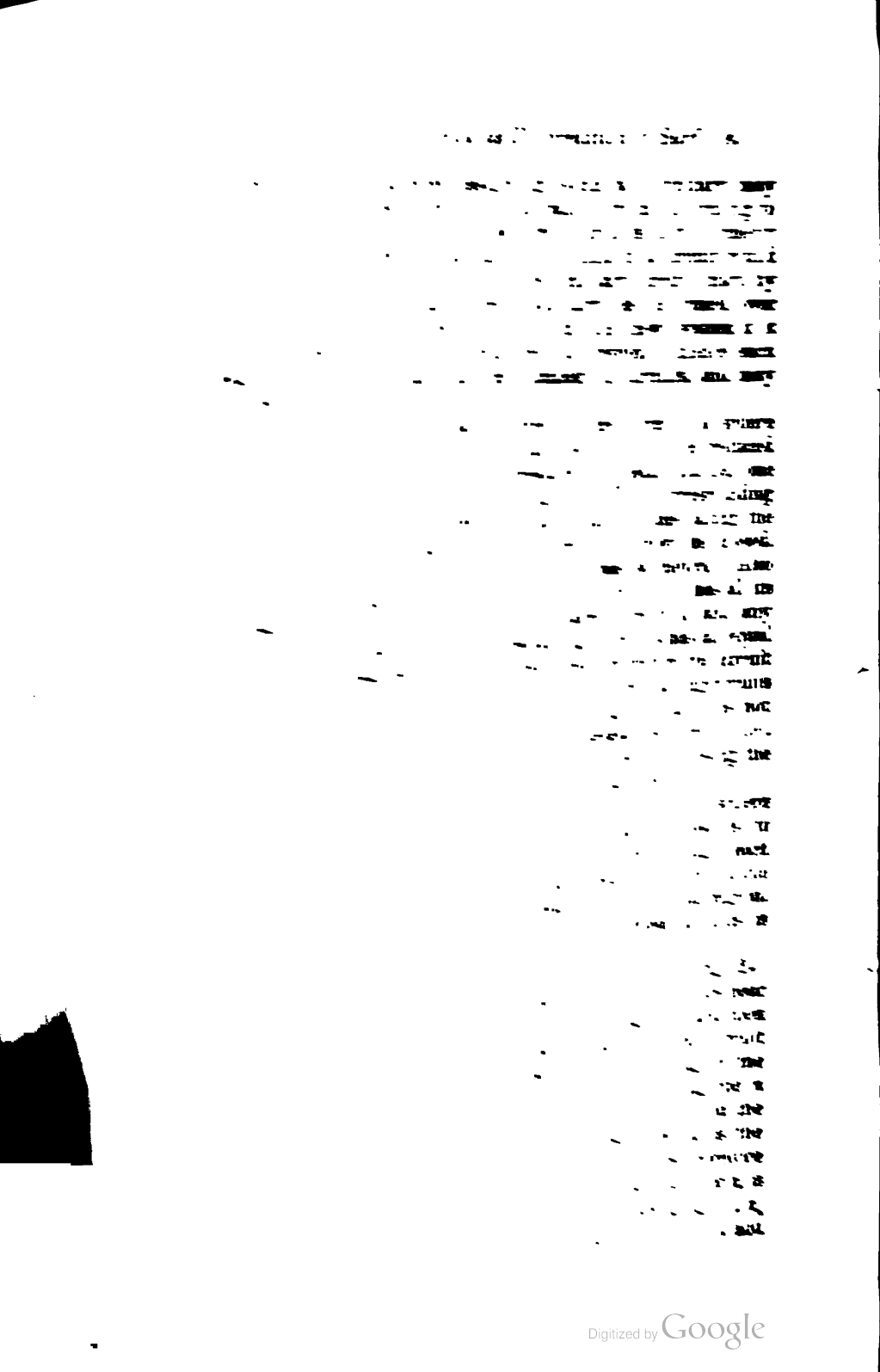

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more than one irreducible circuit to the system, the system is irreducible.

Hence, making a hole or point-boundary on a closed unifacial surface adds one to the number of circuits in a complete irreducible system.

Cor. On a doubly connected closed unifacial surface there is no circuit *irreconcileable* with a zero circuit. A circuit with turning indicator cannot be deformed to evanescence, *i.e.* is *irreducible*, but a circuit consisting of this circuit doubled can be so deformed, that is, the circuit is *reconcileable* with a zero circuit.

23. On a closed unifacial surface a simple complete irreducible system of circuits is not possible.

Consider the surface (of connectivity $n+1$) as a sphere with n unifacial regions, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be circuits one through each of these regions. The circuit $2\alpha_1$ by addition of $2\alpha_2$, $2\alpha_3$, ..., $2\alpha_n$ becomes reducible, that is, the positive directions of the circuits $\alpha_1, \ldots, \alpha_n$ being properly chosen,

$$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n = 0.$$

No circuit passing an even number of times through a unifacial region can be deformed so as to have an odd number of passages through the region, so that the system of circuits obtained by omitting one of the circuits $\alpha_1, \ldots, \alpha_n$ is not simple.

Any circuit may be represented by

$$p_1\alpha_1 + p_2\alpha_2 + \ldots + p_n\alpha_n.$$

Let the $n-1$ circuits $C_1, C_2, \ldots, C_{n-1}$ form a simple complete irreducible system where

$$C_r = p_{r1}\alpha_1 + p_{r2}\alpha_2 + \ldots p_{rn}\alpha_n,$$

for $r = 1, 2, \ldots, n-1$. The system being simple, α_1 may be represented by

$$\lambda_1 C_1 + \lambda_2 C_2 + \ldots \lambda_{n-1} C_{n-1},$$

and therefore $\lambda_1 p_{11} + \lambda_2 p_{21} + \ldots \lambda_{n-1} p_{n-1,1}$ is odd, and at least one of the coefficients $p_{11}, p_{21}, \ldots, p_{n-1,1}$ is odd, say p_{11}.

Put $C_1' = C$; $C_r' = C_r$ or $C_r + C_1$ according as p_{r2} is even or odd, for $r = 2, 3, \ldots, n-1$. There results a system which is simple

$$\begin{aligned}
C_1' &= p'_{11}\alpha_1 + p'_{12}\alpha_2 + \ldots\ldots, \\
C_2' &= 2p'_{21}\alpha_1 + p'_{22}\alpha_2 + \ldots\ldots, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
C'_{n-1} &= 2p'_{n-1,1}\alpha_1 + p'_{n-1,2}\alpha_2 + \ldots,
\end{aligned}$$

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The number of Deformations of a Surface to the Standard Form.

24. When the lines of a standard set of cuts are drawn on a surface and the corresponding deformation to the standard surface made, any number of new deformations may be found by deforming the lines of the cuts and then deforming the surface to the standard form. All these deformations will be considered equivalent. Generally, if there be two deformations T_1 and T_2 of a surface T to the standard form S, then if the surface T_1 can, without leaving the base S, be deformed to T_2 so that every pair of corresponding points on T_1 and T_2 coincide, the deformations T_1 and T_2 are called *equivalent;* if this deformation to coincidence is not possible they are called *distinct.*

In the deformation of a bifacial surface to the standard surface, it is supposed that one face of each is chosen as the positive face, and that the positive faces are made to coincide, that is, that the normals to the positive faces lie in the same direction. If this condition is not demanded the number of distinct deformations is doubled. For there are a certain number of deformations in which the positive faces coincide, and the same number in which positive and negative coincide; also if the positive normal coincide with the positive normal to the standard surface, no deformation without leaving the base can bring it to coincide with a negative normal, and therefore no two deformations of the different sets can be equivalent.

With unifacial surfaces there is no such distinction between faces, and if one deformation to the standard surface is known, another, equivalent or distinct, may be obtained by turning it inside out. ("Inside out" is hardly accurate for a unifacial surface, it refers to the spherical part of the standard surface).

Proposition:—If a surface be deformed in two ways to the standard surface, and there be on it the lines AB, CD, EF,... of cuts which resolve the surface to simple connection, and which become on the two deformations A_1B_1, C_1D_1,... and A_2B_2, C_2D_2,... respectively, then the deformations are equivalent if, and not unless, the lines A_1B_1, ... can be deformed, without meeting themselves or one another or the boundary, to coincide with the lines A_2B_2, ... respectively, in direction as well as position.

Let it be possible to deform the lines in this manner. Make the two sets of cuts so that each deformation of the surface is simply connected. Since the lines of the cuts can

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be so deformed, it is possible to deform the whole boundary of the first simply connected piece without leaving the base so that the parts (if any) which are original boundary coincide always with the boundary of the base, and the parts which are opposite edges of any cut remain in contact, and in the final position the edges of any cut P_1Q_1 coincide with the two edges of the corresponding cut P_2Q_2. Therefore the first simply connected piece may be deformed without leaving the base to coincidence with the second, so that points originally contiguous but separated by cuts remain always infinitely near one another in space. The simply connected surfaces may then be made to coincide point with point. And since points separated by cuts remain infinitely near one another in space, the deformation may be effected without making the cuts; that is, the two deformations are equivalent.

Conversely, if the surfaces are equivalent, the deformation by which the first comes to coincidence with the second gives a deformation of the first set of lines to the second.

If in the foregoing reasoning the standard surface S and the other T be interchanged, it appears that if one set of cuts on S becomes by two deformations two sets on T which cannot be deformed in the above manner to one another, then these two deformations of S to T (or of T to S) are distinct. In the following, the cuts on S will be supposed the standard set, and the number of independent sets on T which can become the standard set on S will be considered.

25. Let the surface with m boundary lines and connectivity n be represented by a sphere with m simple holes, p handles, and q unifacial regions; where if the surface is bifacial $q=0$, $p=\frac{1}{2}(n-m)$: and if the surface is unifacial $q=1, p=\frac{1}{2}(n-m-1)$ or $q=2, p=\frac{1}{2}(n-m-2)$ according as $n-m$ is odd or even.

Proposition:—The number of distinct deformations of a surface to the standard form is infinite if $p>0$, or if $m+q>3$.

Let $p>0$, that is, let there be a handle. Then consider the crosscut which becomes A_1B_1 in the standard set on the standard surface (fig. 19). A line may be drawn passing any number of times r through the handle, and having its ends on the same boundary line, or if there is no boundary line forming a closed curve.

If the surface is unifacial let the line pass once through a unifacial region, so that the corresponding circuit turns the indicator. Then if there is a boundary line, on which the ends of this line lie, the surface may be deformed to the

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standard figure, so that this line coincides with the cut A_1B_1. If the surface is closed a small loopcut may be made on any part of the closed line, and the loopcut and crosscut will then become on the standard figure the loopcut and the crosscut A_1B_1.

Also any deformation of the line passing r times through the handle introduces passages through the handle in pairs in opposite directions, so that the algebraic sum r of the number of passages cannot be altered. Now r may have any integral value, positive or negative, so that the number of distinct ways of making the cut is infinite, and therefore the number of distinct deformations to the standard surface is infinite.

Again, let $m+q>3$. Since q is not greater than 2, there may be

four holes A, B, C, D,
three holes A, B, C, and a unifacial region D,
or two holes A, B, and two unifacial regions C, D.

Consider the crosscut which becomes in the standard figure C_1D_1, joining the chief hole to another. Any crosscut between two boundary-lines may be brought to this position.

Now any loop with its ends fixed and not passing through a unifacial region cannot be deformed so as to pass once through it, or so as to lie on the other side of it, and such loop cannot be deformed across a hole so as to lie on the other side of it. Hence, if C and D are two holes, or two unifacial regions, or a hole and a unifacial region, they prevent, just like a handle, the alteration by deformation of the number of passages of a line between them. Let C and D be holes or unifacial regions, and A and B two holes, then a crosscut from A to B may be drawn passing any number r times between C and D. The number of passages cannot be altered by deformation, and r may have any integral value; the number of independent crosscuts which become C_1D_1 is infinite, and therefore the number of deformations of the surface to the standard form is infinite.

26. It remains to consider surfaces for which $p=0$ and $m+q=$ or <3.

(i) Bounded bifacial surfaces: $p=0$, $q=0$.

If $m=1$ the surface is simply connected, and any two deformations are equivalent.

If $m=2$ the standard set of cuts consists of one C_1D_1 from the chief boundary to the other. And when the chief boundary line is chosen any two cuts are equivalent. For, consider the surface as the part of a circular cylinder cut out by two planes

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perpendicular to its axis, and let $C_1'D_1'$ be any crosscut from the chief boundary to the other. Let D_1' move so as to bring as much as possible of the cut near D_1' to lie along a generator; it will bring the whole line to coincide with a generator, and the line $C_1'D_1'$ may be brought to coincide with any generator, so that all such cuts are equivalent. And the chief hole may be chosen in two ways, so that there are two deformations of the surface to the standard form.

If $m=3$, take the above cylinder with a small hole in its side to represent the surface, and let the small hole be the chief boundary, the upper and lower ends the first and second other boundaries. The standard set of crosscuts consists of two from the chief boundary, C_1D_1 to the first boundary and C_2D_2 to the second. Any cuts $C_1'D_1'$ and $C_2'D_2'$ on the cylinder from the small hole to the upper and lower ends form a standard set; for the only condition is that C_1' and C_2' lie in a certain order round the hole, and there being only two of them they must lie in this order. Draw any line $C_1'D_1'$ and let D_1' move round to cause as much of the line as possible to lie along a generator. It can only be stopped by coming to the hole, as $D_1'K$ (fig. 26). If $C_1'K$ now contract to be as short as possible, C_1' must come to K so that the whole cut lies along a generator through the small hole. Similarly the other cut $C_2'D_2'$ may be brought to lie along the other part of the same generator. When the holes are chosen, there is therefore only one way of deforming the surface to the standard form. And the holes may be chosen in $3\times2\times1$ ways, therefore there are six deformations.

(ii) Bounded unifacial surfaces.

First, $m=1$, $q=1$. The standard figure has one hole with a twisted strip across, and a cut A_1B_1 across the strip is the standard set. Consider the surface as a disc with a unifacial region on it. Any crosscut not dividing the surface may be deformed to run from a fixed point in a straight line to the unifacial region, once through the region, and in a straight line to another fixed point. Therefore all not-dividing cuts are equivalent, in a direction as well as position. And there is only one deformation to the standard surface.

$m=2$, $q=1$. The standard figure has a chief hole with a twisted strip and another simple hole. The crosscuts are A_1B_1 across the strip and C_1D_1 to the other hole. When the chief boundary of the surface is chosen, cuts $A_1'B_1'$, $C_1'D_1'$ form a standard set if $A_1'B_1'C_1'$ lie on the chief boundary in a certain order, and D_1' on the other boundary, and $A_1'B_1'$

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makes the surface bifacial. The order of $A_1'B_1'C_1'$ is always satisfied since there are only three points, and there is no distinction of directions along the boundary line.

Take a disc with a hole and a unifacial region to represent the surface. Draw any line $C_1'D_1'$ joining the boundary lines. Any windings around the hole may be undone by moving D_1' round, and any windings round the hole and unifacial region together may be undone by moving C_1' round. Loops through the unifacial region may then be taken out till the line has one passage through it or none. $C_1'D_1'$ may then be drawn as a straight line through the unifacial region or not, that is, there are two distinct ways of making the cut. When this cut is made the surface is deformable to a disc with a unifacial region. A part of the boundary consists of the edges of the cut and the small hole, and this part must therefore not be passed by the end of any other cut which is being deformed. Draw any crosscut $A_1'B_1'$ making the surface bifacial. If deformation over $C_1'D_1'D_1'C_1'$ were allowed all such cuts would be equivalent, therefore any cuts distinct from $A_1'B_1'$ may be found by considering deformations over this part. The deformation of one end over $C_1'D_1'D_1'C_1'$ interchanges A_1' and B_1', so that another deformation will restore them and there are two distinct ways of making the cut $A_1'B_1'$.

Hence, there are two ways of choosing the chief hole, two ways of making the cut $C_1'D_1'$, and two ways of making the cut $A_1'B_1'$; and therefore there are eight deformations to the standard form.

Lastly $q = 2$, $m = 1$. Take as the surface a disc with two unifacial regions. Any crosscut with both ends on the boundary line may be one of a standard set if the corresponding circuit turns the indicator, that is, if the crosscut have an odd number of passages through unifacial regions. From A_1' (fig. 27) on the boundary draw a line through the two unifacial regions to E and then through the first unifacial region to P: from P let it pass any number of times through the two regions together to Q, R, ..., and finally to B_1' on the boundary. The line has an odd number of passages through unifacial regions and the cut $A_1'B_1'$ may be made to coincide with the cut A_1B_1 on the standard figure.

Consider the part of the line KL from one unifacial region through the other, and then back to the first or to the boundary. The points K, L are confined to the unifacial region or the boundary and therefore the loop KL cannot be deformed so as not to pass through the unifacial region. And therefore the whole line $A_1'B_1'$ cannot be deformed so as to get rid

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In the case $q = 2$, the standard cuts on the standard surface are a loopcut $A_1A_2B_1B_2$ cutting out the simply connected conical piece, a crosscut A_1B_1 with ends on the loopcut across the untwisted strip, and a crosscut A_2B_2 with ends on the loopcut across the twisted strip.

Take as the surface a sphere with two unifacial regions. Any standard set of cuts may be supposed drawn as follows. Draw any loopcut-line L_1 with turning indicator, that is, having an odd number of passages through unifacial regions. Make a small loopcut L_2 inclosing a part of the line L_1, and then a crosscut K_1 along the part $A_1'B_1'$ of L_1 which lies outside L_2. On L_2 take points A_2', B_2' separated by A_1' and B_1', and make a cut K_2 from A_2' to B_2' resolving the surface to simple connection.

Consider the line L_1. If it have two passages in succession through a unifacial region, it may be deformed to get rid of both, so that no line passing twice in succession through the same unifacial region need be considered. Let the line begin at E (fig. 30), and pass first through P of the unifacial regions P, Q to F. It must then return to E, or go through Q to G. In the latter case it cannot return direct from G to E, for the circuit would not then turn the indicator; it must therefore go through P to H. From H it must not return to E, for that gives two successive passages through P; or if from H it goes through Q it must return to the region in which E lies by passing again through P, that is, after any number of passages through Q and P it returns from P to E. This gives two successive passages through P and is not allowable. Therefore from F the line must not go through Q but return immediately to E, and any loopcut-line L_1 is equivalent to a line with only one passage through a unifacial region. This gives two distinct forms for L_1. And the positive and negative descriptions of one of these forms are not equivalent. For if they are, the circuit formed by subtracting one from the other must be reducible: now this circuit consists of one of the lines doubled, and is equivalent to a circuit round the unifacial region, that is, to a circuit passing once between P and Q; and this circuit has been shewn irreducible. There are therefore four distinct ways of making the cut $A_1'B_1'$ and the loopcut $A_1'A_2'B_1'B_2'$ (fig. 31), the direction of the loopcut not considered. The direction of the loopcut is settled by the position of A_2' from which the second crosscut starts. The cut $A_2'B_2'$ alone makes the original surface bifacial and therefore passes through both P and Q; and its ends lie in two regions which are connected only by passage through a uni-

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facial region. Consider the case shewn in fig. 31, and let the cut start at A_2'. When the cut first goes through P it cannot have a successive passage through P and must go direct to B_2'. It must also go through Q but cannot make two successive passages, and so goes from A_2' through Q and P to B_2'. Therefore any cut from A_2' to B_2' is equivalent to this cut. Again, a cut in the opposite direction is equivalent to this cut. For, let the set of lines be deformed so that the loopcut $A_1'A_2'B_1'B_2'$ goes along $B_1'A_1'$ through P and returns to its position: the loopcut is turned over and the crosscuts lie as in fig. 32, $A_1'B_1'$ being unaltered. Let the part of $A_2'B_2'$ from A_2' twice through P to Q be deformed so as to go direct from A_2' to Q, and let the part from Q to P be deformed over the antipodal region of the sphere: the figure thus becomes equivalent to fig. 31. Hence, when the cut $A_1'B_1'$ is chosen there is only one way of making the cut $A_2'B_2'$. And since there are four distinct ways of making the cut $A_1'B_1'$, there are four distinct deformations of the surface to the standard form.

27. All surfaces have now been treated, and since the connectivity is given in terms of p, q, m by $2p+q+m$, or when $m=0$ by $2p+q+1$, the number of deformations of a surface to the standard form is given by the following scheme.

Kind of Surface	Connectivity	Number of Boundary lines	Number of Deformations
Bifacial.....	.....1.....	.....0.....	1
Bifacial.....	.....1.....	.....1.....	1
Bifacial.....	.....2.....	.....2.....	2
Bifacial.....	.....3.....	.....3.....	6
Unifacial....	.....2.....	.....0.....	1
Unifacial....	.....2.....	.....1.....	1
Unifacial....	.....3.....	.....0.....	4
Unifacial....	.....3.....	.....2.....	 8
All other surfaces.....			.....∞

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28. The number of deformations of a surface A to another B (of the same kind, connectivity, and number of boundaries) is the same as the number of deformations to the standard surface S.

Choose one particular way of deforming B and S to one another, and let any number of deformations of A to S be deformed by this way from S to B. Any sets of lines which are equivalent on S will be equivalent on every intermediate form and therefore also on B, and any sets which are distinct on S will be distinct on B. On A draw any standard set of crosscuts. The number of deformations of A to S is the number of distinct forms these cuts can assume on S; which is the same as the number of distinct forms on B. Therefore the number of deformations of A to B is the same as the number of deformations of A to S.

ON THE NINE-POINTS CIRCLE OF A SPHERICAL TRIANGLE.

By Professor Cayley.

THE definition is in effect given in Hart's paper "Extension of Terquem's theorem respecting the circle which bisects three sides of a triangle," *Quarterly Mathematical Journal*, t. IV. (1861), pp. 260—261, viz. if we have a spherical triangle ABC, then we have a circle (*i.e.* a small circle of the sphere), say the nine-point circle, meeting the sides BC, CA, AB in the points F, L; G, M; H, N respectively, where

$$\tan \tfrac{1}{2}BF = \frac{\cos \tfrac{1}{2}b - \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a}{\cos \tfrac{1}{2}c \sin \tfrac{1}{2}a},$$

$$\tan \tfrac{1}{2}CF = \frac{\cos \tfrac{1}{2}c - \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b}{\cos \tfrac{1}{2}b \sin \tfrac{1}{2}a},$$

(which equations agree with $BF + FC = BC$), and

$$\tan \tfrac{1}{2}BL = \frac{\cos \tfrac{1}{2}c \sin \tfrac{1}{2}a}{\cos \tfrac{1}{2}b + \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a},$$

$$\tan \tfrac{1}{2}CL = \frac{\cos \tfrac{1}{2}b \sin \tfrac{1}{2}a}{\cos \tfrac{1}{2}c + \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b},$$

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original triangle: viz. for this triangle $A'B'C'$, we construct the points F', G', H' and L', M', N', and then on the sides of the triangle ABC taking $BF=2BF'$, $BL=2BL'$, &c., we have the points F, G, H, L, M, N.

Thus p, q, r, p_1, q_1, r_1 denoting as above the cosines and sines of the half-sides of the triangle ABC, that is the cosines and sines of the sides of the triangle $A'B'C'$, we have

$$\tan B'F' = \frac{q-rp}{rp_1}, \quad \tan C'F' = \frac{r-pq}{qp_1},$$

$$\vdots$$

$$\tan B'L' = \frac{rp_1}{q+rp}, \quad \tan C'L' = \frac{qp_1}{r+pq}.$$

$$\vdots$$

First, for the points F', G', H', we have $A'F'$, $B'G'$ $C'H'$, the perpendiculars from the angles on the opposite sides, meeting in a point O', the orthocentre of the triangle $A'B'C'$: in fact, from the triangle $A'B'F'$, right-angled at F', we have

$$\tan B'F' = \tan A'B' \cos B' = \tan \tfrac{1}{2}c \, \frac{\cos \frac{1}{2}b - \cos \frac{1}{2}c \cos \frac{1}{2}a}{\sin \frac{1}{2}c \sin \frac{1}{2}a},$$

$$= \frac{r_1}{r} \frac{q-rp}{r_1 p_1} = \frac{q-rp}{rp_1}$$

as above, and similarly for the points G' and H'.

I notice that we have

$$\sin B'F' = \frac{q-rp}{\sqrt{\{(q-rp)^2 + r^2 p_1^2\}}}, = \frac{q-rp}{\sqrt{(q^2+r^2-2pqr)}},$$

and thus

$$\sin B'F' = \frac{q-rp}{\sqrt{(q^2+r^2-2pqr)}}, \quad \sin C'F' = \frac{r-pq}{\sqrt{(q^2+r^2-2pqr)}},$$

$$\sin C'G' = \frac{r-pq}{\sqrt{(r^2+p^2-2pqr)}}, \quad \sin A'G' = \frac{p-qr}{\sqrt{(r^2+p^2-2pqr)}},$$

$$\sin A'H' = \frac{p-qr}{\sqrt{(p^2+q^2-2pqr)}}, \quad \sin B'H' = \frac{q-rp}{\sqrt{(p^2+q^2-2pqr)}};$$

hence

$$\sin B'F' . \sin C'G' . \sin A'H' = \sin C'F' . \sin A'G' . \sin B'H',$$

which (as is well known) is the condition for the intersection of the arcs $A'F'$, $B'G'$, $C'H'$ in the orthocentre O'.

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viz. the arcs AL, BM, CN meet in a point, which is obviously *not* the cos-centre of the triangle ABC.

We have thus the construction of the nine-points circle as a six-points circle, by means of the points F, G, H, L, M, N; and by way of recapitulation we may say that the nine-points circle meets the sides BC, CA, AB in the points F, L; G, M; H, N respectively, where the points F, G, H depend on the ortho-centre of the semi-triangle, and the points L, M, N depend on the cos-centre of the semi-triangle.

The triangle ABC has an inscribed circle and three escribed circles, and we have (as is known) the theorem that the nine-points circle touches each of these four circles. The circles BC, CA, AB and the nine-points circle form a tetrad of circles, and the inscribed circle and the three escribed circles a tetrad of circles, or say the eight circles form a bitetrad, such that each circle of the one tetrad touches each circle of the other tetrad.

LIST OF THE TRANSITIVE SUBSTITUTION GROUPS OF TEN AND OF ELEVEN LETTERS.

By Dr. F. N. Cole, Ann Arbor.

I. THE determination of the transitive groups of ten letters is a somewhat lengthier process than for nine letters, owing mainly to the increased number of possibilities in the non-primitive forms, but otherwise it presents hardly any new difficulty of a serious kind. It is impossible within the space that I am disposed to claim here to give any account of minor methods employed in individual cases, and accordingly the following contains beside a summary of results only the most indispensable landmarks, sufficient however, I believe, for checking the final list.

For the non-primitive groups the ten letters divide into two sets of five or five sets of two. In the former case the intransitive bases (which must affect the two sets of letters symmetrically) can only be the following:—

Order.	
14400.	$(abcde)$ all $(fghij)$ all,
7200.	$\{(abcde)$ all $(fghij)$ all$\}$ pos.,
3600.	$(abcde)$ pos. $(fghij)$ pos.,
400.	$(abcde)_{00}\,(fghij)_{00}$

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200. $\{(abcde)_{20}(fghij)_{20}\}$ pos.,

120. $(abcde.fghij)$ all,

100. 1 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_1$,

2 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_2$,

3 $(abcde)_{10}(fghij)_{10}$,

60. $(abcde.fghij)$ pos.,

50. $\{(abcde)_{10}(fghij)_{10}\}$ dim.,

25. $(abcde)$ cyc. $(fghij)$ cyc.,

20. $(abcde.fghij)_{20}$,

10. $(abcde.fghij)_{10}$,

5. $(abcde.fghij)$ cyc.

To each of the above intransitive groups are to be added substitutions which (1) transform them into themselves, which (2) interchange the two systems of intransitivity, and whose (3) squares are contained in the given intransitive group. In this way the following 29 groups are generated:—

28800. $(abcde)$ all $(fghij)$ all $(af.bg.ch.di.ej)$,

14400. 1 $\{(abcde)$ all $(fghij)$ all$\}$ pos. $(af.bg.ch.di.ej)$,

2 $\{(abcde)$ all $(fghij)$ all$\}$ pos. $(afbg.ch.di.ej)$,

7200. $(abcde)$ pos. $(fghij)$ pos. $(af.bg.ch.di.ej)$,

800. $(abcde)_{20}(fghij)_{20}(af.bg.ch.di.ej)$,

400. 1 $\{(abcde)_{20}(fghij)_{20}\}$ pos. $(af.bg.ch.di.ej)$,

2 $\{(abcde)_{20}(fghij)_{20}\}$ pos. $(afbg.ch.di.ej)$,

240. *$(abcde.fghij)$ all $(af.bg.ch.di.ej)$,

200. 1 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_1$ $(af.bg.ch.di.ej)$,

2 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_1$ $(afbg.chej.di)$,

3 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_1$ $(afbgdich.ej)$,

4 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_2$ $(af.bg.ch.di.ej)$,

5 $\{(abcde)_{20}(fghij)_{20}\}$ quad.$_2$ $(afbg.chej.di)$,

6 $(abcde)_{10}(fghij)_{10}(af.bg.ch.di.ej)$,

7 $(abcde)_{10}(fghij)(afbg.chej.di)$,

8 $(abcde)_{10}(fghij)_{10}(ajbh.cfeg.di)$,

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120. 1 *$(abcde.fghij)$ pos. $(af.bg.ch.di.ej)$,

2 *$(abcde.fghij)$ pos. $(af.bhei.cjdg)$,

100. 1 $\{(abcde)_{10}(fghij)_{10}\}$ dim. $(af.bg.ch.di.ej)$

2 $\{(abcde)_{10}(fghij)_{10}\}$ dim. $(afbg.chej.di)$,

3 $\{(abcde)_{10}(fghij)_{10}\}$ dim. $(ajbh.cfeg.di)$,

50. 1 $(abcde)_5(fghij)_5(af.bg.ch.di.ej)$,

2 $(abcde)_5(fghij)_5(af.bh.cj.dg.ei)$,

3 $(abcde)_5(fghij)_5(af.bj.ci.dh.eg)$,

40. *$(abcde.fghij)_{20}(af.bg.ch.di.ej)$,

20. 1 *$(abcde.fghij)_{10}(af.bg.ch.di.ej)$,

2 *$(abcde.fghij)_{10}(agdh.bicf.ej)$,

10. 1 *$(abcde.fghij)_5(af.bg.ch.di.ej)$,

2 *$(abcde.fghij)_5(af.bj.ci.dh.eg)$.

The groups marked * in the preceding list are non-primitive in both ways, having not only two systems of five letters but also five systems of two letters.

Of intransitive bases suitable for bases of non-primitives of the second type, five systems of two letters, I find only

32. $(ab)(cd)(ef)(gh)(ij)$,

16. $\{(ab)(cd)(ef)(gh)(ij)\}$ pos.

2. $(ab.cd.ef.gh.ij)$,

and identity. The non-primitives arising from the last two cases I find to be all included in the preceding table, being those marked there with *. The first two cases give rise to the following 13 forms:—

3840. $(ab)(cd)(ef)(gh)(ij)\left\{\begin{matrix}(acegi.bdfhj)\\(ace.bdf)\\(ac.bd)\end{matrix}\right\}$,

1920. 1 $(ab)(cd)(ef)gh)(ij)\left\{\begin{matrix}(acegi.bdfhj)\\(ace.bdf)\end{matrix}\right\}$,

2 $\{(ab)(cd)(ef)(gh)(ij)\}$ pos. $\left\{\begin{matrix}(acegi.bdfhj)\\(ace.bdf)\\(ac.bd)\end{matrix}\right\}$,

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$$8 \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ace.bdf)\\(ac.bd.ef)\end{matrix}\right\},$$

$$960. \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ace.bdf)\end{matrix}\right\},$$

$$640. \quad (ab)(cd)(ef)(gh)(ij) \left\{\begin{matrix}(acegi.bdfhj)\\(ceig.dfjh)\end{matrix}\right\},$$

$$320. \quad 1 \quad (ab)(cd)(ef)(gh)(ij) \left\{\begin{matrix}(acegi.bdfhj)\\(ci.eg.fh.dj)\end{matrix}\right\},$$

$$2 \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ceig.dfjh)\end{matrix}\right\},$$

$$3 \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ceig.dfjh.ab)\end{matrix}\right\},$$

$$160. \quad 1 \quad (ab)(cd)(ef)(gh)(ij)(acegi.bdfhj),$$

$$2 \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ci.eg.dj.fh)\end{matrix}\right\},$$

$$3 \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } \left\{\begin{matrix}(acegi.bdfhj)\\(ci.eg.dj.fh.ab)\end{matrix}\right\},$$

$$80. \quad \{(ab)(cd)(ef)(gh)(ij)\} \text{ pos. } (acegi.bdfhj).$$

In a primitive group of ten letters the subgroup which leaves a single letter unchanged must affect all the nine remaining letters and can only be one of the following (where the notation is the same as in my list of the groups of nine letters):—

Order: $3, 6_2, 6_3, 9_2, 9_3, 9_4, 12_{14}, 18_4, 18_{10}, 18_{11}, 18_{12}, 18_{13}, 27_2, 27_3, 36_9, 36_{10}, 36_{11}, 54_6, 54_7, 54_8, 54_9, 72_{13}, 72_{14}, 72_{15}, 144_{14}, 216_8, 432_3, 504, 1512, \frac{1}{2}9!, 9!$.

The corresponding primitive groups of ten letters, if such exist, have for their orders these numbers multiplied by ten.

Several cases can be immediately rejected. There can be no primitive groups of ten letters of degrees 30, 90, 180, 270, 540, 2160 or 4320. For a group of order 30 always contains a cyclical subgroup of order 15*, which in the present case

* Cf. *Amer. Jour. Math.*, Vol. XV. pp., 215 seq. Also Hölder: *Math. Ann.*, Bd. 43, p. 412.

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must contain circular substitutions of three letters. Also a group of order 90 contains, by Sylow's* theorem, either 1 or 6 subgroups of order 5 each transformed into itself by 90 or 15 substitutions, so that there would be in either case a (cyclical) subgroup of order 15. Similar considerations dispose of the remaining orders. (In the last case the positive substitutions would form a (non-existent) subgroup of order 2160).

Also, in case of the orders 6_9, 36_9, and 36_{10}, the positive substitutions of the corresponding primitive groups of ten letters would form primitive groups of orders 30 and 180, which have just been shewn to be non-existent.

The orders 5040 and 15120 are also to be rejected. Each of the corresponding groups, as is readily found, must contain 120 subgroups of order 7. In the former case every subgroup of order 7 is transformed into itself by a subgroup of order 42. The latter is transitive in seven letters but not in eight, nine, or ten. On trial it is found impossible to construct such a subgroup which can be contained in a primitive group. In the second case a subgroup of order 7 is transformed into itself by a subgroup of order 126. The latter contains either 1 or 7 subgroups of order 9, and in the second case in therefore isomorphic with a transitive group of 7 letters of as order which is a divisor of 126, and is therefore 7, 14, 21, or 42. The group of order 126 therefore contains a self-conjugate subgroup of order 18, 9, 6, or 3. In any case some substitution of order 3 must be transformed into itself by a substitution of order 7, and then will therefore be a cyclical substitution of order 21 in the group, which is here impossible.

There now remain only the following base-groups in nine letters:—

$$6_8,\ 12_{14},\ 36_{11},\ 72_{13},\ 72_{14},\ 72_{15},\ 1444_{14},\ \tfrac{1}{2}9\,!,\ 9\,!,$$

and each of these, I find, gives rise to a single primitive type in ten letters.

The groups of order 60 and 120 are readily seen to contain each 6 subgroups of order 5, and to be isomorphic with the alternating and symmetric groups respectively of five letters. These groups must therefore contain some substitution with two cycles of five letters each of which is transformed into its fourth power by $ad.bf.ce.gh$. On trial the only admissible form for this substitution is found to be

$$iaf\,bd.jehgc.$$

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* *Math. Ann.*, Bd. v, pp. 584—94.

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The two groups of orders 60 and 120 are accordingly

60. $(abc.def.ghi)$ cyc. $(ad.bf.ce.gh)$ $(iafbd.jehgc)$ cyc.,

120. $G_{60}(ad.be.cf)$.

The group of order 360 must contain 36 subgroups of order 5, each of which is transformed into itself by a subgroup of order 10. Since the group is even and therefore contains no cyclical substitution of ten letters, it appears that there is some substitution with two cycles of five letters each which is transformed into its fourth power by $ac.bd.eg.fh$. I find essentially only one such substitution, otherwise permissible, viz.

$$iaegc.jhbdf,$$

and the only primitive group of order 360 is therefore

360. $(abcd.efgh)$ cyc. $(iac.bgh.dfe)$ cyc. $(iaegc.jhdbf)$ cyc.

Two of the primitive groups of order 720 are half positive and half negative. These therefore contain the preceding type as a subgroup. The third case also present no serious difficulty. I find for these three groups

720. 1 $G_{360}(ac.ef.gh)$,

2 $G_{360}(afbgchde)$ cyc.,

3 $G_{360}(ahcf.bgde)$ cyc.

Finally the primitive group of order 1440 contains the last case as its positive subgroup. Accordingly we have

1440. $G_{720_2}(ac.ef.gh)$.

To these are to be added the alternating and symmetric groups

$\frac{1}{2}10!$ $(abcdefghij)$ all,

$10!$ $(abcdefghij)$ pos.

The primitive groups of order 60 and 120 are simply transitive. Those of orders 360 and 720_1 are two-fold, and those of orders 720_2, 720_3, and 1440 three-fold transitive.

II. A transitive group of eleven letters is primitive, and therefore contains no substitutions of two, three, four, five, or seven letters. From a theorem of Jordan*, its order is a divisor of $4.6.8.9.10.11 = 11.17280$. Furthermore, such a group must contain subgroups of order 11, and by Sylow's

* Jordan: *Traité des substitutions*, p. 665.

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theorem the number of the latter is of the form $11k+1$, and is a divisor of the order of the group, the quotient being the number of substitutions in the group which transform a subgroup of order 11 into itself. Now the only divisors of 11.17280 which are of the form $11k+1$ are

$$1,\ 12,\ 45,\ 144,\ 320,\ 540,\ 1728.$$

Again, the symmetric group of eleven letters contains $9!$ subgroups of order 11, each of which is therefore transformed into itself by $\frac{11!}{9!} = 110$ substitutions of eleven letters.

Accordingly the order of a transitive group of eleven letters is of the form

$$11\alpha(11k+1),$$

where α is a divisor of 10, and $11k+1$ is one of the seven numbers above. Of these orders those can be at once rejected for which $\alpha = 1$. For, in this case, of the $11(11k+1)$ substitutions of the group $10(11k+1)$ would be of order 11 and only $11k+1$ would remain. But the subgroup which leaves a single one of the eleven letters unchanged is itself of order $11k+1$, and the eleven subgroups of this kind would contain more than $11k+1$ substitutions, unless indeed $k=0$.

The possible orders of transitive groups of eleven letters are therefore included among the following sixteen:—

11,	11.10,	11.90,	11.640,	11.1440,
11.2,	11.24,	11.120,	11.720,	11.3456,
11.5,	11.60,	11.288,	11.1080,	11.8640,
		11.17280.		

I consider now first the simply transitive groups. Of these those whose orders are divisible by 5, contain substitutions of order 5 which must consist of two cycles of five letters each. Accordingly the (intransitive) subgroups which affect only two letters each have two systems of intransitivity of five letters each. No substitutions of the five letters of the one system can occur with two different substitutions of the letters of the second system. For then a proper combination would give substitutions affecting the second system but not the first. Consequently the subgroup affecting ten letters rests on a one-to-one isomorphism between two transitive groups of five letters each, and its order is 120, 60, 20, 10 or 5. The orders 120 and 20 are to be rejected since the

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group (*abcde.fghij*) *all* contains substitutions of four letters, and the table above does not include 11.20. Also, the group (*abcde.fghij*) *pos.* contains 10 conjugate subgroups of order 3 affecting six letters each, and therefore the corresponding group of eleven letters would contain $\frac{11}{6} \times 10 = 22$ such groups, each of which would be transformed into itself by a group of order $\frac{6.6.0}{2.2} = 30$. The latter must contain a cyclical subgroup of order 15 which is here inadmissible.

Hence, the order of a simply transitive group of eleven letters which contains a substitution of order 5 *can be only* 110 *or* 55.

For the remaining orders

11, 11.2, 11.24, 11.288, 11.3456,

it is to be noticed that if a group of order 11 is transformed into itself by 22 substitutions, 11 of these are of order 2 and consist of 5 cycles of 2 letters each. The corresponding transitive group of eleven letters is therefore half odd. Its even substitutions form a self-conjugate subgroup of half the order of the original group. But a self-conjugate subgroup of a primitive group is transitive.* Now, as there are no simply transitive groups of eleven letters of orders 11.12, 11.144, 11.1728, there are none of orders 11.24, 11.288, 11.3456.

A simply transitive group of eleven letters is of order 11, 22, 55, *or* 110.

For each of these orders there can be only one subgroup of order 11, which is therefore transformed into itself by the entire group. The latter is therefore in each case a subgroup of Kronecker's well-known metacyclic group.† The group of order 110 is doubly transitive.

The orders of multiply transitive groups of eleven letters are divisible by 110. The only orders in the table which satisfy this condition and which can have transitive subgroups of ten letters, with no substitutions of less than six letters, are‡

11.10, 11.60, 11.120, 11.720, 11.1440.

The group of order 110 is the metacyclic group already mentioned. Those of orders 11.120 and 11.1440 contain substitutions with 5 cycles of two letters, and are therefore half odd. The problem is, then, to determine those of the two orders 11.60 and 11.720.

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* Netto: *Theory of Substitutions*, p. 82.
† *Ibid*, p. 152.
‡ *V.* the preceding list of transitive groups of ten letters.

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For the first of these two cases the subgroup which leaves a letter unchanged can only be the primitive group of order 60 in ten letters. For the second case the same subgroup must be even (otherwise the group would contain a self-conjugate, and therefore transitive, even subgroup of order 360), and there is only one even transitive group of order 720 in ten letters.

It is known that there is a group of order 660 in eleven letters, *viz.* the simple group of this order identified with the theory of the transformation of the 11th order of the elliptic modular functions.* And Matthieu† has constructed a four-fold transitive group of eleven letters, which being composed of even substitutions, must from the preceding reasoning be of order 720. It is therefore sufficient to show that there is only one group of each of these orders.

The group of order 660 contains 12 subgroups of order 11, each of which is therefore transformed into itself by 55 substitutions of the group. These form a semi-metacyclic group, which when the subgroup of order 11 is given can be found in only one way. We may assume this semi-metacyclic group to be

$$(abcdefghijk) \text{ cyc. } (bdjfe.cghki) \text{ cyc.}$$

Again, the group of order 660 contains 66 subgroups of order 5, each transformed into itself by a non-cyclical subgroup of order 10. The 5 substitutions of order 2 in the latter, being even, affect 8 elements. For the subgroup $(bdjfe.cghki)$ cyc. the substitution of order 2 which leaves the letter b unchanged must be one of the 5 following :—

$$de.jf.gi.hk, \quad de.jf.ch.ik, \quad de.jf.ci.gk,$$
$$de.jf.cg.hi, \quad de.jf.ck.gh.$$

But the products of $abcdefghijk$ into the first, second, and fourth of these are substitutions which cannot occur in the group. Also, the third case is transformed into the fifth by the substitution

$$bk.cj.di.eh.fg$$

which transforms the group of order 55 above itself. Accordingly we obtain the same result essentially whether we combine the third or the fifth substitution of order 2 with the group of order 55.

This combination leads to a transitive group of an order as

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* Cf. Klein-Fricke : *Modulfunctionen.*
† Matthieu : *Liouville* (2), t. 5 and 6.

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large as $11.5.2 = 110$, which is not the metacyclic group, and can therefore only be the group, known to exist, of order 660.

There is only one type of group of order 660 *in eleven letters, viz., the simple group of this order, connected with the theory of the elliptic modular-functions. It may be written*

$(abcdfghijk)$ cyc. $(bdjfe.cghki)$ cyc. $(de.jf.ck.gh)$.¶

One of the subgroups of order 60 is readily found to be

$(bdjfe.cghki)$ cyc. $(fge.hjd.bck)$ cyc.

To this must be added, to form a primitive group of order 120 in the 10 letters, the substitution

$$bi.dg.eh.cjkf.$$

But the latter does not transform the group of order 660 into itself.

There is therefore no group of order 11.120 *in eleven letters.*

The group of order 11.720 contains 144 subgroups of order 11, each transformed into itself, as in the case of the group of order 660, by 55 substitutions of the group. There are, however, now 11.36 subgroups of order 5, each transformed into itself by 20 substitutions. We must now add to the group

$(abcdefghijk)$ cyc. $(bdjfe.cghki)$ cyc.

a substitution with 2 cycles of 4 letters each which transforms $bdjfe.cghki$ into its second power, and the square of which is $de.jf.ck.gh$. This can only be

$$djef.cgkh$$

and this in combination with the group of order 55 generates a group of an order as large as 220, which is not contained in the group of order 660, and which is therefore the Matthieu group of order 11.720.

There is one and only one group of order 11.720 *in eleven letters, viz.,*

$(abcdefghijk)$ cyc. $(bdjfe.cghki)$ cyc. $(djef.cgkh)$ cyc.*

Transforming this group with respect to the substitution $ak.eh.jb.fd$, we obtain

$(kjcfhdgeiba)$ cyc. $(jfbdh.cgeai)$ cyc. $(fbhd.cgae)$ cyc.

$= (abiegdhfcjk)$ cyc. $(iaegc.jhdbf)$ cyc. $(aecg.bhdf)$ cyc.

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* This group like that of order 660 is *simple.*

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Let

$$abiegdhfcjk = \rho,\ iaegc.jhdbf = \sigma,\ aecg.dfbh = \tau;$$

we have

$$\rho\sigma^3 = adf.big.cjk,\ \rho^{-1}(\rho\sigma^3)\rho^7 = fbe.cja.gdh = s,$$

$$s^{-1}\sigma s = icfdj.agheb = t,\ tst^{-1} = ceh.idb.afg.$$

Accordingly, in the group of order 11.720, the subgroup which does not affect the letter k contains the substitutions

$$iaegc.jhdbf,\ aecg.bhdf,\ ibd.che.agf.$$

Also

$$\rho^3\tau\rho^{-3} = bdjg.ichf,\ \sigma^3(\rho^3\tau\rho^{-3}) = iehd.abgc,$$

$$idb.ceh.afg \times iehd.abgc = afch.bedg,$$

so that the subgroup mentioned also contains $afch.hedg$, and is therefore (*cf.* the list of group of ten letters) actually the positive transitive group of order 720 in the letters.

If now there is any group of order 11.1440 in eleven letters, it must be obtained by adding to that just found the substitution

$$ac.ef.gh.$$

This transforms the subgroup of order 720, just considered, into itself. But the substitution ρ becomes

$$\rho' = ajkcbifhdge;$$

also

$$\rho\rho'^3 = afijbhdge,$$

which does not occur in the subgroup of order 720.

There is therefore no group of eleven letters of order 11.1440.

Following is a complete list of the transitive groups of eleven letters:—

Order.	
11 !	(*abcdefghijk*) all,
½11 !	(*abcdefghijk*) pos.,
7920.	$\left\{\begin{matrix}(abiegdhfcjk)\text{ cyc. }(abcd.efgh)\text{ cyc.}\\(iaegc.jhdbf)\text{ cyc. }(ahcf.bgde)\text{ cyc.}\end{matrix}\right.$ $(iac.bgh.dfe)$ cyc.$\Big\}$
	= (*abiegdhfcjk*) cyc. (*iaegc.jhdbf*) cyc. (*abcd.efgh*) cyc.,
660.	(*abiegdhfcjk*) cyc. (*iaegc.jhdbf*) cyc. (*ac.eg.bd.fh*),
110.	(*abcdefghijk*) cyc. (*bceifkjhdg*) cyc.,
55.	(*abcdefghijk*) cyc. (*bdjfe.cghki*) cyc.,
22.	(*abcdefghijk*) cyc. (*bk.cj.di.eh.fg*),
11.	(*abcdefghijk*) cyc.

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Of these the third is four-fold, the fourth and fifth are two-fold, and the last three are simply transitive.

I avail myself of this opportunity to note an error in my corrected list of the groups of eight letters in the preceding volume of this *Journal*. On p. 374 at the top two of the six groups of order 16 are identical, viz.

1,	$ac.bd$,	1,	$eg.fh$,	1,	$ac.bd$,	1,	$eg.fh$,
ac,	bd,	$efgh$,	$ehgf$,	ac,	bd,	$ef.gh$,	$eh.fg$,
$ab.cd$,	$ad.bc$,	eg,	fh,	$ab.cd$,	$ad.bc$	$efgh$,	$ehgf$,
$abcd$,	$adcb$	$ef.gh$,	$eh.fg$,	$abcd$,	$adbc$,	eg,	fh.

At the bottom of p. 373 the last eight lines should be omitted.

Ann Arbor.
Jan, 1894.

THE MOTION OF A SATELLITE ABOUT A SPHEROIDAL PLANET.

By F. W. Dyson, M.A., Fellow of Trinity College, Cambridge; Isaac Newton Student in the University of Cambridge.

The Motion of a Satellite round a Spheroidal Planet.

IN this paper the motion of a spherical satellite about a spheroidal planet is discussed. Equations in the canonical form are obtained for the motion of the planet about its centre by the method of Jacobi and Hamilton. These are transformed, giving six equations for determining the elements of the planet's motion about its centre. The six equations giving the elements of the satellite's orbit are quoted from Planetary Theory. The disturbing function is expanded in terms of the time: the secular part of it being obtained carefully for any spheroid whose surfaces of equal density consist of similar spheroids. The secular inequalities are considered fully: it is found that the mean distance between the bodies and the inclination of the planet's axis to its axis of resultant momentum are unaltered and that the motion of the node of the plane of resultant momentum of the planet and of the Satellite's orbit on the invariable plane of the system is nearly uniform. The periodic terms are considered

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which occur in the Moon's Librations. The mean distance and the inclination of the planet's axis to its axis of resultant momentum are shewn to be stable when terms of the second order in the disturbing masses are considered.

Section I.

§ 1. *The equations for the variation of the elements of the motion of an oblate spheroid about its centre of gravity.*

A spheroid under no forces spins uniformly round its axis, while that axis describes a cone round the line of resultant angular momentum. To determine the motion completely it is necessary to know:

The position of the line of resultant angular momentum:

The inclination of the axis of the spheroid to this line, and its position at one definite time:

The angular velocity of the spheroid about its axis, and the position of a given meridian at this definite time.

Altogether, *six* elements must be known. The action of disturbing forces on the spheroid will cause these to vary. We have to find the equations giving the variation of these elements. We shall first integrate the equations of the undisturbed motion by the method of Hamilton and Jacobi, as this leads readily to the formation of the required equations. The motion of any body under no forces has been considered in this way by M. Serret (*Mémoires de l' Académie des Sciences*, t. XXXV), and by M. Radau (*Annales de l' Ecole Normale Supérieure* 1re *Série* t. VI). It is, however, easier to find the equations independently for the simpler case of a solid of revolution.

§ 2. Let A be the position (fig. 34) of the line of resultant angular momentum of the spheroid: P its axis of figure: N the descending node of the polar of A on the plane of reference: O the point from which longitudes are measured.

Let θ and ψ be the latitude and longitude of P. Then $OM = \psi$; $MP = \theta$. Let ϕ be Euler's third coordinate, that is, the angle which a plane through the axis fixed in the spheroid makes with the plane ZPM.

Let C and A be the moments of inertia of the spheroid round its axis and a perpendicular diameter, respectively.

The kinetic energy is given by

$$2T = A(\dot{\theta}^2 + \cos^2\theta\dot{\psi}^2) + C(\dot{\phi} + \dot{\psi}\sin\theta)^2.$$

Let the momenta corresponding to θ, ϕ, ψ be Θ, Φ, Ψ.

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$$\left.\begin{aligned} \text{Then } \Theta &= \frac{\partial T}{\partial \dot\theta} = A\dot\theta, \\ \Phi &= \frac{\partial T}{\partial \dot\phi} = C(\dot\phi + \dot\psi \sin\theta), \\ \Psi &= \frac{\partial T}{\partial \dot\psi} = A\cos^2\theta\dot\psi + C\sin\theta(\dot\phi + \dot\psi\sin\theta) \end{aligned}\right\}.$$

$$\text{Therefore } 2T = \frac{1}{A}\left\{\Theta^2 + \sec^2\theta(\Psi - \Phi\sin\theta)^2\right\} + \frac{1}{C}\Phi^2,$$

so that Hamilton's principal function S satisfies the equation

$$2\frac{\partial S}{\partial t} + \frac{1}{A}\left\{\left(\frac{\partial S}{\partial\theta}\right)^2 + \sec^2\theta\left(\frac{\partial S}{\partial\psi} - \sin\theta\frac{\partial S}{\partial\phi}\right)^2\right\} + \frac{1}{C}\left(\frac{\partial S}{\partial\phi}\right)^2 = 0.*$$

We require an integral of this equation containing three arbitrary constants.

Put $S = -\alpha_1 t + \alpha_2\phi + \alpha_3\psi + S_1$, where S_1 is independent of t, ϕ, and ψ.

$$\text{Then } -2\alpha_1 + \frac{\alpha_2^2}{C} + \frac{1}{A}\left\{\left(\frac{\partial S_1}{\partial\theta}\right)^2 + \sec^2\theta(\alpha_3 - \alpha_2\sin\theta)^2\right\} = 0.$$

Writing, for brevity,

$$\alpha_4^2 = A\left(2\alpha_1 - \frac{\alpha_2^2}{C}\right),$$

this equation gives

$$S = -\alpha_1 t + \alpha_2\phi + \alpha_3\psi + \int_{\theta_1}^{\theta}\frac{\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2\}}}{\cos\theta}\,d\theta.$$

The lower limit is arbitrary: let it be taken to be the least root of the equation

$$\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2 = 0.$$

§ 3. The integrals of the dynamical equations for the motion of the spheroid may be taken as

$$\left.\begin{aligned} \beta_1 &= \frac{\partial S}{\partial\alpha_1}, \\ \beta_2 &= \frac{\partial S}{\partial\alpha_2}, \\ \beta_3 &= \frac{\partial S}{\partial\alpha_3} \end{aligned}\right\},$$

where β_1, β_2, β_3 are three additional constants.

* Routh, *Rigid Dynamics*, Vol. II., p. 257.

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Since $\alpha_4 \cos^2\theta_1 - (\alpha_3 - \alpha_2 \sin\theta_1)^2 = 0$, these equations are

$$\beta_1 = -t + A\int_{\theta_1}^{\theta} \frac{\cos\theta\, d\theta}{\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2\}}},$$

$$\beta_2 = \phi - \frac{A}{C}\alpha_2 \int_{\theta_1}^{\theta} \frac{\cos\theta\, d\theta}{\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2\}}}$$
$$+ \int_{\theta_1}^{\theta} \frac{\sin\theta\,(\alpha_3 - \alpha_2\sin\theta)\, d\theta}{\cos\theta\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2\}}},$$

$$\beta_3 = -\psi - \int_{\theta_1}^{\theta} \frac{(\alpha_3 - \alpha_2\sin\theta)\, d\theta}{\cos\theta\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2\}}}.$$

These equations give θ, ϕ, ψ in terms of t and the six arbitrary constants α_1, α_2, α_3 β_1, β_2, β_3. These quantities are *Canonical elements*, and lead very readily to the equations of the disturbed motion.

§ 4. We now proceed to find what these constants are in terms of the elements of the motion mentioned in § 1.

Let $\frac{1}{2}\pi - \alpha - \beta$ and $\frac{1}{2}\pi - \alpha + \beta$ be the roots of the equation

$$\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2 = 0.$$

It is easily seen that

$$\frac{\alpha_2}{\cos\beta} = \frac{\alpha_3}{\cos\alpha} = \frac{\alpha_4}{\sin\beta}.$$

Let each of these fractions $= x$. Then

$$\alpha_1 = \tfrac{1}{2}x^2\left(\frac{\cos^2\beta}{C} + \frac{\sin^2\beta}{A}\right),$$

$$\alpha_2 = x\cos\beta,$$

$$\alpha_3 = x\cos\alpha,$$

$$\alpha_4 = x\sin\beta.$$

The quantity under the square root

$$\alpha_4^2\cos^2\theta - (\alpha_3 - \alpha_2\sin\theta)^2$$

must therefore be always positive.

Thus $\{\cos(\frac{1}{2}\pi - \beta - \theta) - \cos\alpha\}\{\cos\alpha - \cos(\frac{1}{2}\pi - \theta + \beta)\}$ must be always positive.

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Therefore θ must be $< \frac{1}{2}\pi - \alpha + \beta$,

and $> \frac{1}{2}\pi - \alpha - \beta$.

It follows that β is the radius of the small circle described by the pole about the line of resultant angular momentum, and $\frac{1}{2}\pi - \alpha$ is the latitude of this line.

Thus, in (fig. 34),

$$ZA = \alpha \text{ and } AP = \beta,$$

and $$\theta_1 = RP_1 = \tfrac{1}{2}\pi - \alpha - \beta.$$

Let N be the descending node of the great circle parallel to the small circle described by P.

Let $ON = N$; then $OR = N + \frac{1}{2}\pi$.

Let ϕ_1 be the value of ϕ when P is at P_1, *i.e.* when $\theta = \theta_1$.

The equations of § 3, shew that

$$\beta_2 = \phi_1,$$

and $$\beta_3 = N + \tfrac{1}{2}\pi.$$

Let the angle $P_1AP = \chi$,

$$\sin\theta = \cos ZP$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\chi.$$

Therefore $\cos\theta\, d\theta = \sin\alpha\sin\beta\sin\chi\, d\chi$.

Also $\sqrt{\{\alpha_4^2\cos^2\theta - (\alpha_1 - \alpha_2\sin\theta)^2\}} = x\sin\alpha\sin\beta\sin\chi$.

Therefore,

$$\beta_1 = -t + \frac{A}{x}\chi,$$

or $$\chi = \frac{x}{A}(t + \beta_1).$$

Therefore $\frac{x}{A} = n_1 =$ angular velocity of P in its circle

$$= \frac{C}{A}\sec\beta,$$

or $$x = C\sec\beta.$$

Also, if τ be the time when P is at P_1,

$$\beta_1 = -\tau.$$

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We have, thus, the following canonical system,

$$\alpha_1 = \tfrac{1}{2}C^2\omega^2 \sec^2\beta \left(\frac{\sin^2\beta}{A} + \frac{\cos^2\beta}{C}\right), \qquad \beta_1 = -\tau,$$

$$\alpha_2 = C\omega, \qquad \beta_2 = \phi_1,$$

$$\alpha_3 = C\omega \sec\beta \cos\alpha, \qquad \beta_3 = N + \tfrac{1}{2}\pi.$$

If ε_1 = longitude of P when $t = 0$,

$$\varepsilon_1 - N - \tfrac{1}{2}\pi = \frac{C}{A}\omega \sec\beta . \beta_1$$

$$= -\frac{C}{A}\omega \sec\beta . \tau.$$

Therefore, instead of $-\tau$, we may write

$$\frac{A}{C\omega \sec\beta}(\varepsilon_1 - N - \tfrac{1}{2}\pi).$$

§ 5. Solving these equations, we have

$$C\omega = \alpha_2, \qquad \varepsilon_1 = \beta_3 + \frac{\beta_1}{A}\sqrt{\left\{\frac{C-A}{C}\alpha_2^2 + 2A\alpha_1\right\}},$$

$$\tan^2\beta = 2\frac{A\alpha_1}{\alpha_2^2} - \frac{A}{C}, \qquad \phi_1 = \beta_2,$$

$$\cos\alpha = \frac{\alpha_3}{\sqrt{\left\{\frac{C-A}{C}\alpha_2^2 + 2A\alpha_1\right\}}}, \quad N = \beta_3 - \tfrac{1}{2}\pi.$$

Let R be the disturbing function: then R is expressible as a function of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, t$: or of α, β, ω, &c., t; and we have the equations

$$\frac{\partial R}{\partial \alpha_1} = \frac{A}{C^2\omega^2}\cos^2\beta \left\{\cot\alpha \frac{\partial R}{\partial \alpha} + \cot\beta \frac{\partial R}{\partial \beta} + (\varepsilon_1 - N - \tfrac{1}{2}\pi)\frac{\partial R}{\partial \varepsilon_1}\right\},$$

$$\frac{\partial R}{\partial \alpha_2} = \frac{1}{C\omega}\left\{\omega \frac{\partial R}{\partial \omega} - \frac{A}{C}\cot\beta\left(\frac{\sin^2\beta}{A} + \frac{\cos^2\beta}{C}\right)\frac{\partial R}{\partial \beta}\right.$$

$$\left. + \frac{C-H}{C}\cot\alpha\cos^2\beta \frac{\partial R}{\partial \alpha} + \frac{C-A}{C}\cos^2\beta(\varepsilon_1 - N - \tfrac{1}{2}\pi)\frac{\partial R}{\partial \varepsilon_1}\right\},$$

$$\frac{\partial R}{\partial \alpha_3} = -\frac{1}{C\omega}\frac{\cos\beta}{\sin\alpha}\frac{\partial R}{\partial \alpha},$$

$$\frac{\partial R}{\partial \beta_1} = \frac{C}{A}\omega \sec\beta \frac{\partial R}{\partial \varepsilon_1},$$

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$$\frac{\partial R}{\partial \beta_2} = \frac{\partial R}{\partial \phi_1},$$

$$\frac{\partial R}{\partial \beta_3} = \frac{\partial R}{\partial \varepsilon_1} + \frac{\partial R}{\partial N}.$$

Since $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are a *canonical system*,

$$\frac{d\alpha_1}{dt} = \frac{\partial R}{\partial \beta_1}, \qquad \frac{d\beta_1}{dt} = -\frac{\partial R}{\partial \alpha_1},$$

$$\frac{d\alpha_2}{dt} = \frac{\partial R}{\partial \beta_2}, \qquad \frac{\partial \beta_2}{dt} = -\frac{\partial R}{\partial \alpha_2},$$

$$\frac{d\alpha_3}{dt} = \frac{\partial R}{\partial \beta_3}, \qquad \frac{d\beta_3}{dt} = -\frac{\partial R}{\partial \alpha_3}.$$

Finding $\frac{d\beta}{dt}$, &c. in terms of $\frac{d\alpha_1}{dt}$, $\frac{d\alpha_2}{dt}$, &c., and substituting the above values of $\frac{\partial R}{\partial \beta_1}$, &c., we obtain

$$C\,\frac{d\omega}{dt} = \frac{\partial R}{\partial \phi_1},$$

$$C\omega\,\frac{d\phi_1}{dt} = -\left\{\omega\,\frac{\partial R}{\partial \omega} - A\cot\beta\left(\frac{\sin^2\beta}{A} + \frac{\cos^2\beta}{C}\right)\frac{\partial R}{\partial \beta} + \frac{C-A}{C}\cot\alpha\cos^2\beta\,\frac{\partial R}{\partial \alpha} + \frac{C-A}{C}\cos^2\beta\,(\varepsilon_1 - N - \tfrac{1}{2}\pi)\,\frac{\partial R}{\partial \varepsilon_1}\right\},$$

$$C\omega\,\frac{\sin\alpha}{\cos\beta}\cdot\frac{d\alpha}{dt} = -\frac{\partial R}{\partial N} - (1-\cos\alpha)\,\frac{\partial R}{\partial \varepsilon_1} + \cos\alpha\cos\beta\,\frac{C-A}{C}\cdot\frac{\partial R}{\partial \phi_1},$$

$$C\omega\,\frac{\sin\alpha}{\cos\beta}\cdot\frac{dN}{dt} = \frac{\partial R}{\partial \alpha},$$

$$C\omega\,\frac{\cos^2\beta}{\sin\beta}\cdot\frac{d\beta}{dt} = \frac{\partial R}{\partial \varepsilon_1} - A\sec\beta\left(\frac{\sin^2\beta}{A} + \frac{\cos^2\beta}{C}\right)\frac{\partial R}{\partial \phi_1},$$

$$\frac{C\omega}{\cos\beta}\cdot\frac{d\varepsilon_1}{dt} = \frac{1-\cos\alpha}{\sin\alpha}\,\frac{\partial R}{\partial \alpha} - \cot\beta\,\frac{\partial R}{\partial \beta} + \frac{C-A}{C}\cos\beta\,(\varepsilon_1 - N - \tfrac{1}{2}\pi)\,\frac{\partial R}{\partial \phi_1}.$$

Let us write $\frac{C}{A}\,\omega\sec\beta = n_1$.

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When the perturbing function is developed, it will be found that ω only occurs through n_1: while n_1 only occurs in series and cosines of multiples of $n_1 t + \varepsilon_1$.

β occurs both through n_1 and otherwise.

Denoting, henceforward, by $\frac{\partial R}{\partial \beta}$ the differential coefficient of R as if n_1 were an independent variable: and taking a new ε_1 as in planetary theory, so that

$$\int n_1 dt + \varepsilon_1 = n_1 t + \varepsilon_1,$$

we find

$$C\frac{d\omega}{dt} = \frac{\partial R}{\partial \phi_1},$$

$$A\frac{dn_1}{dt} = \frac{\partial R}{\partial \varepsilon_1} + \frac{C-A}{C}\cos\beta\frac{\partial R}{\partial \phi_1},$$

$$\frac{d}{dt}(An_1\cos\alpha) = \frac{\partial R}{\partial N} + \frac{\partial R}{\partial \varepsilon_1},$$

$$An_1\sin\alpha\frac{dR}{dt} = \frac{\partial R}{\partial \alpha},$$

$$An_1\frac{d\phi_1}{dt} = -\frac{C-A}{C}\cot\alpha\cos\beta\frac{\partial R}{\partial \alpha} + A\operatorname{cosec}\beta$$

$$\times\left(\frac{\sin^2\beta}{A} + \frac{\cos^2\beta}{C}\right)\frac{\partial R}{\partial \beta} - \frac{C-A}{C}\cos\beta\left(\int n_1 dt + \varepsilon_1 - N - \tfrac{1}{2}\pi\right)\frac{\partial R}{\partial \varepsilon_1},$$

$$An_1\frac{d\varepsilon_1}{dt} = \frac{1-\cos\alpha}{\sin\alpha}\frac{\partial R}{\partial \alpha} - \cot\beta\frac{\partial R}{\partial \beta}$$

$$+ \frac{C-A}{C}\cos\beta\left(\int n_1 dt + \varepsilon_1 - N_1 - \tfrac{1}{2}\pi\right)\frac{\partial R}{\partial \phi_1}.$$

§ 6. These are the equations determining the variation of the elements of the spheroid's motion about its centre, due to forces which have a force-function R.

If these forces arise from the attraction of external matter, and if the spheroid be absolutely rigid, and accurately a solid of revolution, R will not involve ϕ_1: so that the above equations may be simplified by putting $\frac{\partial R}{\partial \phi_1} = 0$: and the first of them may be integrated giving $\omega = \text{const.}$ This, however, does not hold if the spheroid yields at all, or if there are any

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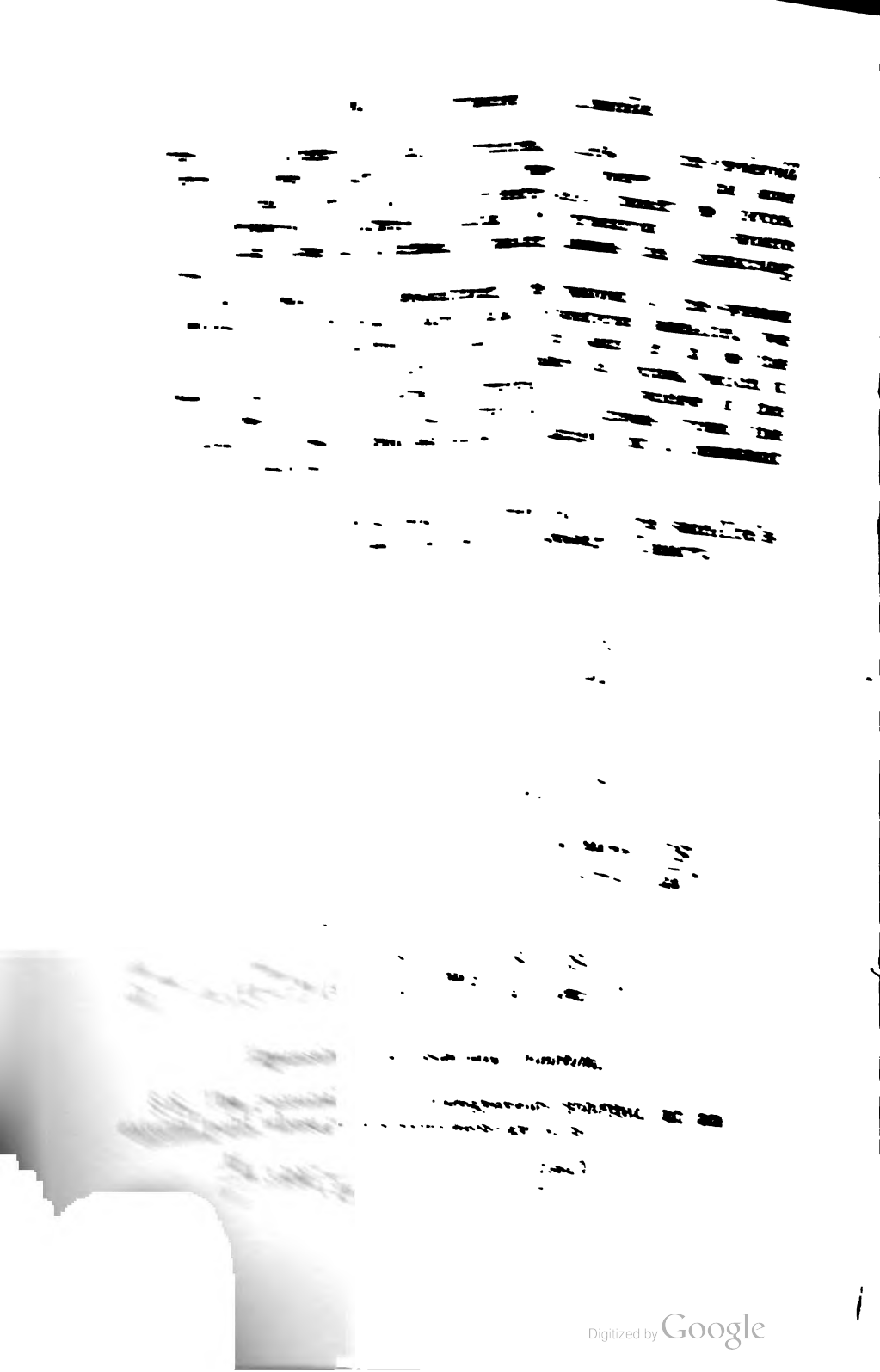

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where M is the mass of the spheroid and c^2 the difference of the squares of its semi-axes.

If the spheroid is not homogeneous, but its mass is distributed symmetrically about its centre, the potential will be of the form

$$\frac{M}{r} + 3M\Sigma_1^\infty A_p \frac{P_{2p}(\cos\vartheta)}{r^{2p+1}},$$

where A_p is a constant depending on the distribution of the mass of the spheroid.

In particular, the first terms will be

$$\frac{M}{r} + \frac{C-A}{r^3} P_2(\cos\vartheta),$$

where C and A are the moments of inertia about the axis and an equatorial diameter respectively.

The disturbing function R to be used in considering the motion of the spheroid about its centre is

$$3Mm\Sigma_1^\infty A_p \frac{P_{2p}(\cos\theta)}{r^{2p+1}}.$$

And the disturbing function to be used in considering the motion of the satellite is

$$3(M+m)\Sigma_1^\infty A_p \frac{P_{2p}(\cos\theta)}{r^{2p+1}}.$$

We shall write $\Sigma_1^\infty A_p \frac{P_{2p}(\cos\theta)}{r^{2p+1}} = F.$

We proceed now to develop F in terms of the elements of the motion of the planet and satellite.

§ 9. Let $O\Omega N$ be the plane of reference (fig. 34). Let Z be its pole; O the point from which longitudes are measured.

Let $\Omega K A_0 Q$ be the plane of the satellite's orbit: Ω its ascending node on the plane of reference: A_0 the satellite's position at perigee, and Q at the time t.

Let $KNRL$ be the plane of resultant angular momentum of the planet: A its pole: K the ascending node of the plane of the satellite's orbit upon it. Let R be 90° in front of K.

Let P be the position of the planet's pole at time t.

Let $O\Omega = \Omega$, $ON = N$, $ON + NL = \lambda$, $O\Omega + \Omega Q = \theta$,

$$O\Omega + \Omega A_0 = \varpi.$$

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Let $\angle K\Omega N = i$, $\angle KN\Omega = a$.

Also, let $N - KN = N'$,

$\Omega + \Omega K = \Omega'$,

and $\angle A_0 KN = j$.

Then $$\left.\begin{aligned} NL &= \lambda - N, & KL &= \lambda - N' \\ \Omega A_0 &= \varpi - \Omega, & KA_0 &= \varpi - \Omega' \\ \Omega Q &= \theta - \Omega, & KQ &= \theta - \Omega' \end{aligned}\right\}.$$

Now $$\cos PQ = \cos PA \cos QA + \cos PK \cos QK + \cos PR \cos QR,$$

$\cos PA = \cos\beta$, $\cos QA = \sin j \sin(\theta - \Omega')$,

$\cos PK = \sin\beta \cos(\lambda - N')$, $\cos QK = \cos(\theta - \Omega')$,

$\cos PR = \sin\beta \sin(\lambda - N')$, $\cos QR = \cos j \sin(\theta - \Omega')$.

Therefore

$$\cos PQ = \cos\beta \sin j \sin(\theta - \Omega') + \sin\beta \cos(\lambda - N') \cos(\theta - \Omega') + \sin\beta \cos j \sin(\lambda - N') \sin(\theta - \Omega').$$

This is a function of $\beta, j, \lambda - N', \theta - \varpi, \varpi - \Omega'$.
The disturbing function is in addition a function of r.
The expressions of the form

$$\frac{\sin k(\theta - \varpi)}{r^{2p+1}}, \quad \frac{\cos k(\theta - \varpi)}{r^{2p+1}},$$

which will occur, may be expanded as functions of

$$a,\ e,\ nt + \varepsilon - \varpi.$$

Thus, writing $n_1 t + \varepsilon_1$ instead of λ, we find that the disturbing function F is a function of

$$\beta, j, a, e, \varpi - \Omega',\ nt + \varepsilon - \varpi,\ n_1 t + \varepsilon_1 - N'.$$

§ 10. The case which will be considered is the one in which Ω and N coincide: so that $\Omega' = \Omega$, $N' = N$.

In finding $\frac{\partial F}{\partial \Omega}$, $\frac{\partial F}{\partial N}$, in this case we cannot put $\Omega' = \Omega$ till after the differentiation.

Denoting by $\left(\frac{\partial F}{\partial \Omega}\right)$, $\left(\frac{\partial F}{\partial N}\right)$ the expressions we should

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obtain by differentiating after Ω' has been put $= \Omega$ and $N' = N$, we find

$$\frac{\partial F}{\partial \Omega} = \left(\frac{\partial F}{\partial \Omega}\right)\frac{\partial \Omega'}{\partial \Omega} + \left(\frac{\partial F}{\partial N}\right)\frac{\partial N'}{\partial \Omega} + \frac{\partial F}{\partial j}\frac{\partial j}{\partial \Omega},$$

and

$$\frac{\partial F}{\partial N} = \left(\frac{\partial F}{\partial \Omega}\right)\frac{\partial \Omega'}{\partial N} + \left(\frac{\partial F}{\partial N}\right)\frac{\partial N'}{\partial N} + \frac{\partial F}{\partial j}\frac{\partial j}{\partial N}.$$

Now $\cos j = \cos\alpha \cos i - \sin\alpha \sin i \cos(N - \Omega)$.

Therefore $\dfrac{\partial j}{\partial \Omega} = 0$ and $\dfrac{\partial j}{\partial N} = 0$

when $N = \Omega$, and also $j = \alpha + i$.

Also $\Omega' = \Omega + \Omega K$,

$N' = N - KN$;

and, considering ΩKN as a plane triangle, which is allowable when it is very small,

$$\Omega K = \frac{\sin\alpha}{\sin(\alpha + i)}(N - \Omega),$$

$$NK = \frac{\sin i}{\sin(\alpha + i)}(N - \Omega).$$

Therefore

$$\frac{\partial F}{\partial \Omega} = +\left\{1 - \frac{\sin\alpha}{\sin(\alpha + i)}\right\}\left(\frac{\partial F}{\partial \Omega}\right) + \frac{\sin i}{\sin(\alpha + i)}\left(\frac{\partial F}{\partial N}\right),$$

and

$$\frac{\partial F}{\partial N} = +\frac{\sin\alpha}{\sin(\alpha + i)}\left(\frac{\partial F}{\partial \Omega}\right) + \left\{1 - \frac{\sin i}{\sin(\alpha + i)}\right\}\left(\frac{\partial F}{\partial N}\right).$$

Again, since F is $F(n_1 t + \varepsilon_1 - N,\ nt + \varepsilon - \varpi,\ \varpi - \Omega)$,

$$\left.\begin{aligned} \frac{\partial F}{\partial \varepsilon_1} + \left(\frac{\partial F}{\partial N}\right) &= 0 \\ \frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \varpi} + \left(\frac{\partial F}{\partial \Omega}\right) &= 0 \end{aligned}\right\}.$$

Therefore, if we replace in the formulæ

$$\left.\begin{aligned} \frac{\partial F}{\partial N} \text{ by } &-\frac{\sin\alpha}{\sin(\alpha + i)}\left(\frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \varpi}\right) - \left(1 - \frac{\sin i}{\sin(\alpha + i)}\right)\frac{\partial F}{\partial \varepsilon_1} \\ \text{and } \frac{\partial F}{\partial \Omega} \text{ by } &-\left(1 - \frac{\sin\alpha}{\sin(\alpha + i)}\right)\left(\frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \varpi}\right) - \frac{\sin i}{\sin(\alpha + i)}\frac{\partial F}{\partial \varepsilon_1} \end{aligned}\right\},$$

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we may take F, when N and Ω coincide in its simplified form, as a function of

$$a, e, i, \alpha, \beta, nt + \varepsilon - \varpi, n_1 t + \varepsilon_1 - N, \varpi - N.$$

§ 11. The disturbing function is given, as far as the first power of the ellipticity of the spheroid is concerned, by the equation

$$F = A_1 \frac{P_2(\cos PQ)}{r^3}.$$

Substituting the value of $\cos PQ$

$$\begin{aligned}\frac{F}{A_1} r^3 = &- \tfrac{1}{2} P_2(\cos\beta) P_2(\cos j) \\ &+ \tfrac{3}{8} \sin 2\beta \sin 2j \sin(\lambda - N') \\ &+ \tfrac{3}{8} \sin^2\beta \sin^2 j \cos 2(\lambda - N') \\ &- \tfrac{3}{4} P_2(\cos\beta) \sin^2 j \cos 2(\theta - \Omega') \\ &- \tfrac{3}{8} \sin 2\beta \sin 2j \sin(\lambda - N') \cos 2(\theta - \Omega') \\ &+ \tfrac{3}{8} \sin^2\beta (1 + \cos^2 j) \cos 2(\lambda - N') \cos 2(\theta - \Omega') \\ &+ \tfrac{3}{4} \sin 2\beta \sin j \cos(\lambda - N') \sin 2(\theta - \Omega') \\ &+ \tfrac{3}{4} \sin^2\beta \cos j \sin 2(\lambda - N') \sin 2(\theta - \Omega').\end{aligned}$$

In the case when N and Ω coincide, we may put

$$N' = \Omega' = N.$$

To obtain F as a function of the elements and the time, we must expand

$$\frac{1}{r^3}, \frac{\sin 2(\theta - \varpi)}{r^3}, \frac{\cos 2(\theta - \varpi)}{r^3}$$

in terms of the time.

§ 12. *The secular part of the disturbing function for a planet of any oblateness.*

This can be found exactly for all values of the eccentricity and obliquity of the satellite's orbit, as well as for all values of the ellipticity of the planet. Let B be the pole of the orbit.

Join BA, BZ, BQ.

Let S be the pole of BQ (fig. 34).

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The disturbing function

$$F = 3\Sigma_1^\infty A_p \frac{P_{2p}(\cos PQ)}{r^{2p+1}}.$$

But $P_{2p}(\cos PQ) = P_{2p}(\cos PA)\, P_{2p}(\cos QA)$

$$+ 2\Sigma \frac{(2p-k)!}{(2p+k)!} T_{2p}^{(k)}(\cos PA)\, T_{2p}^{(k)}(\cos QA) \cos k(PAQ).$$

Now $$PAQ = PAN + NAQ,$$

and $$PAN = n_1 t + \varepsilon_1 - N.$$

Thus the disturbing function consists of terms of the form

$$CT_{2p}^{(k)}(\cos QA) \frac{1}{r^{2p+1}} \cos k(QAN) \cos k(n_1 t + \varepsilon_1 - N),$$

and $$CT_{2p}^{(k)}(\cos QA) \frac{1}{r^{2p+1}} \sin k(QAN) \sin k(n_1 t + \varepsilon_1 - N).$$

$$T_{2p}^{(k)}(\cos QA),\ \frac{1}{r^{2p+1}},\ \sin k(QAN),$$

all involve t, but only through sines and cosines of multiples of $nt + \varepsilon$.

The above expressions are, thus, of the form

$$\Sigma \left\{ D \cos k'(nt+\varepsilon) + E \sin k'(nt+\varepsilon) \right\} \begin{matrix} \sin \\ \cos \end{matrix} k(n_1 t + \varepsilon_1).$$

If, therefore, n_1 and n are incommensurable—that is, provided there is no commensurable relation between the length of the satellite's year and the period of the pole of the planet round the invariable line—these terms will not give rise to any secular inequalities.

The secular part of the disturbing function is therefore contained in

$$3\Sigma_1^\infty A_p P_{2p}(\cos\beta) \frac{P_{2p}(\cos QA)}{r^{2p+1}}.$$

Again, $P_{2p}(\cos QA) = P_{2p}(\cos BA)\, P_{2p}(\cos BQ)$

$$+ 2\Sigma \frac{(2p-k)!}{(2p+k)!} T_{2p}^{(k)}(\cos BA)\, T_{2p}^{(k)}(\cos BQ) \cos k(QBA).$$

The angle $QBA = KS$

$$= \tfrac{1}{2}\pi - KQ$$

$$= \tfrac{1}{2}\pi - (\theta - \Omega'),$$

$$BQ = \tfrac{1}{2}\pi.$$

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Therefore $T_{2p}^{(k)}(\cos BQ) = 0$, when k is odd.

Also $\cos BA = \cos j$

$$= \cos\alpha\cos i - \sin\alpha\sin i\cos(N-\Omega).$$

Therefore the secular terms of the disturbing function are contained in

$$3\Sigma_1^{\infty}\left\{\frac{(-1)^p.a^{2p}}{(2p+1)(2p+3)}\cdot\frac{P_{2p}(\cos S)}{r^{2p+1}}\left[P_{2p}(0)\,P_{2p}(\cos j)\right.\right.$$

$$\left.\left.+2\Sigma\frac{(2p-2q)!}{(2p+2q)!}T_{2p}^{(2q)}(0)\,T_{2p}^{(2q)}(\cos j)(-1)^q\cos 2q(\theta-\Omega')\right]\right\}.$$

Let $\dfrac{\cos 2q(\theta-\Omega'-k)}{r^{2p+1}}$

$$= A_0 + A_1\cos(nt+\varepsilon) + A_2\cos 2(nt+\varepsilon) + \&c.$$
$$+ B_1\sin(nt+\varepsilon) + B_2\sin 2(nt+\varepsilon) + \&c.$$

Write for brevity, $nt+\varepsilon = \zeta$,

and $\theta - \varpi = w$.

Then $2\pi A_0 = \displaystyle\int_0^{2\pi}\frac{\cos 2q(\theta-\Omega')}{r^{2p+1}}\,d\zeta.$

But $r^2\dfrac{d\theta}{dt} = h = na^2\sqrt{(1-e^2)}.$

Therefore $d\zeta = \dfrac{r^2}{a^2}\dfrac{d\theta}{\sqrt{(1-e^2)}}$,

Also $\dfrac{1}{r} = \dfrac{1}{a}\dfrac{1+e\cos w}{1-e^2}$.

Therefore $2\pi A_0$

$$= \frac{1}{a^{2p+1}}(1-e^2)^{-2p+\frac{1}{2}}\int_0^{2\pi}(1+e\cos w)^{2p-1}\cos 2q(w+\varpi-\Omega')\,dw.$$

Writing $e = \sin\phi$,

$$2\pi A_0 = \frac{\cos 2q(\varpi-\Omega')}{a^{2p+1}(\cos\phi)^{4p-1}}\int_0^{2\pi}(1+\sin\phi\cos w)^{2p-1}\cos 2qw\,dw$$

$$= \frac{\cos 2q(\varpi-\Omega')}{a^{2p+1}\cos^{2p}\phi}\int_0^{2\pi}(\sec\phi+\tan\phi\cos w)^{2p-1}\cos 2qw\,dw.$$

This integral is expansible as a tesseral harmonic.

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Denoting the series

$$\mu^{i-\sigma} - \frac{(i-\sigma)(i-\sigma-1)}{2(2i-1)}\mu^{i-\sigma-2} + \frac{(i-\sigma)(i-\sigma-1)(i-\sigma-2)(i-\sigma-3)}{2.4(2i-1)(2i-3)}\mu^{i-\sigma-4} - \&c.,$$

by the symbol $\mathfrak{P}_\sigma^i(\mu)$,

(Heine, *Handbuch der Kugelfunctionen*, p. 117).

then $$\frac{2^i i!(i-\sigma)!}{(2i)!} T_i^{(\sigma)} = (1-\mu^2)^{\frac{1}{2}\sigma}\mathfrak{P}_\sigma^i(\mu).$$

And $$\frac{1}{2\pi}\int_0^{2\pi}(\sec\phi + \tan\phi\cos w)^{2p-1}\cos 2qw\,dw = \frac{2^{-2p+1}(4p-2)!}{(2p+2q-1)!(2p-2q-1)!}\tan^{2q}\phi\,\mathfrak{P}_{2q}^{2p-1}(\sec\phi).$$

Also $$T_{2p}^{(2q)}(\cos j) = \frac{(4p)!}{2^{2p}(2p)!(2p-2q)!}\sin^{2q}j\,\mathfrak{P}_{2q}^{2p}(\cos j),$$

and $$T_{2p}^{(2q)}(0) = (-1)^{p+q}\frac{(2p+2q)!}{2^{2p}(p+q)!(p-q)!},$$

and $$\frac{1}{2\pi}\int_0^{2\pi}(\sec\phi + \tan\phi\cos w)^{2p-1}dw = P_{2p-1}(\sec\phi) = \frac{2^{-2p+1}(4p-2)!}{\{(2p-1)!\}^2}\mathfrak{P}_0^{2p-1}(\sec\phi).$$

Therefore the secular part of the disturbing function is $3F_0$, where F_0

$$= \Sigma_1^\infty \left\{ A_p \frac{1}{a^{2p+1}} 2P(\cos\beta)\sec^{2p}\phi\,[\mathfrak{P}_0^{2p}(\cos j)\,\mathfrak{P}_0^{2p-1}(\sec\phi) + 2\Sigma B_q \sin^{2q}j\,\mathfrak{P}_{2q}^{2p}(\cos j)\tan^{2q}\phi\,\mathfrak{P}_{2q}^{2p-1}(\sec\phi)\cos 2q(\varpi-\Omega')]\right\},$$

where A_p and B_q are numerical constants, given by

$$B_q = \frac{(p!)^2}{(p+q)!(p-q)!}\,\frac{\{(2p-1)!\}^2}{(2p+2q-1)!(2p-2q-1)!},$$

$$A_p = \frac{1}{2^{6p-1}(2p+1)(2p+3)}\,\frac{(4p!)}{\{2p!\}^2}\,\frac{(4p-2)!}{\{(2p-1)!\}^2}\,\frac{(2p)!}{\{p!\}^2}.$$

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When the obliquity j vanishes, F_0 takes the simpler form

$$3\Sigma_1^\infty C_p \frac{c^{2p}}{a^{2p+1}} P_{2p}(\cos\beta) \sec^{2p}\phi \, P_{2p-1}(\sec\phi),$$

and when the eccentricity $\sin\phi$ vanishes,

$$3\Sigma_1^\infty C_p \frac{c^{2p}}{a^{2p+1}} P_{2p}(\cos\beta) P_{2p}(\cos i);$$

C_p is the same in both cases, its value being

$$\frac{1}{(2p+1)(2p+3)} \frac{(2p)!}{\{2^p.p!\}^2}.$$

§ 13. The principle of conservation of angular momentum shews the general nature of the motion of a satellite round a spheroidal planet, where there is no third body to disturb the motion.

Let ω be the angular velocity of the planet about its axis of figure, C, A its moments of inertia about this axis and an equatorial axis respectively. Let the planet's axis be inclined at an angle β to the axis of resultant angular momentum of the planet. Let OA be this line. The resultant angular momentum of the planet is $C\omega \sec\beta$ about OA (fig. 35).

Let the angular momentum of the translatory motion of the planet and satellite about their centre of gravity be h about the line OB.

Then $$h = \frac{Mm}{M+m} na^2 \sqrt{(1-e^2)},$$

where a and e are the semi-major axis and eccentricity of the satellite's instantaneous orbit about the planet.

Let the resultant of $C\omega \sec\beta$ about OA and h about OB be h_0 about OZ.

Then h_0 is constant and OZ is the invariable line of the system.

Writing $\frac{C\omega}{A} \sec\beta = n_1$, as before,

let $AOZ = \alpha$, $BOZ = i$,

$$\left.\begin{aligned} An_1 \cos\alpha + \frac{Mm}{M+m} na^2 \sqrt{(1-e^2)} \cos i &= h_0 \\ \text{and} \qquad An_1 \sin\alpha - \frac{Mm}{M+m} na^2 \sqrt{(1-e^2)} \sin i &= 0 \end{aligned}\right\},$$

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or
$$\frac{An_1}{\sin i} = \frac{\frac{Mm}{M+m} na^2 \sqrt{(1-e^2)}}{\sin \alpha} = \frac{h_0}{\sin(\alpha + i)}.$$

Also, since OA, OZ, OB are in one plane, the ascending node of the satellite's orbit on the invariable plane coincides with the descending node of the plane of resultant angular momentum for the planet alone.

If the spheroid attracted an external particle as a sphere of the same mass would do, the motion would be simple. N would be a fixed point, α and i would be fixed, β would be fixed: so that the planet would merely spin about its axis, while that axis described a cone uniformly, and the satellite would describe a fixed elliptic orbit about the centre of the planet as focus. When there is a disturbing force a, e, i, α,&c., vary subject to the relations given above.

§ 14. The results of § 13 are easily deducible from the equations for the variations of the elements. Writing $\sin\phi$ for e, we obtain, when the disturbing function is independent of ϕ_1,

$$\frac{da}{dt} = 6na^2 \frac{\partial F}{\partial \varepsilon} \quad \ldots\ldots(1),$$

$$\frac{d}{dt}(na^2 \cos\phi) = 3n^2a^2 \left(\frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \varpi}\right) \quad \ldots\ldots(2),$$

$$\frac{d}{dt}(na^2 \cos\phi \cos i) = \frac{3n^2a^2}{\sin(\alpha+i)} \left\{\sin\alpha \left(\frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \varpi}\right) - \sin i \frac{\partial F}{\partial \varepsilon_1}\right\} \quad \ldots\ldots(3),$$

$$\frac{d\Omega}{dt} = \frac{3na}{\cos\phi \sin i} \frac{\partial F}{\partial i} \quad \ldots\ldots(4),$$

$$\frac{d\varpi}{dt} = 3na \left\{\frac{1}{\sin\phi} \frac{\partial F}{\partial \phi} + \frac{\tan \frac{1}{2} i}{\cos\phi} \frac{\partial F}{\partial i}\right\} \quad \ldots\ldots(5),$$

$$\frac{d\varepsilon}{dt} = -6na^2 \frac{\partial F}{\partial a} + 3na \tan\tfrac{1}{2}\phi \frac{\partial F}{\partial \phi} + 3na \tan\tfrac{1}{2} i \sec\phi \frac{\partial F}{\partial i} \quad \ldots(6),$$

$$\frac{d\omega}{dt} = 0 \quad \ldots\ldots(7),$$

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$$\frac{dn_1}{dt} = 3\frac{Mm}{A}\frac{\partial F}{\partial \varepsilon_1} \quad \ldots\ldots(8),$$

$$\frac{d}{dt}(n_1\cos\alpha) = \frac{3Mm}{A\sin(\alpha+i)}\left\{-\sin\alpha\left(\frac{\partial F}{\partial \varepsilon}+\frac{\partial F}{\partial \varpi}\right)+\sin i\frac{\partial F}{\partial \varepsilon_1}\right\} \quad \ldots\ldots(9),$$

$$\frac{dN}{dt} = \frac{3Mm}{An_1\sin\alpha}\frac{\partial F}{\partial \alpha} \quad \ldots\ldots(10),$$

$$\frac{d\varepsilon_1}{dt} = \frac{3Mm}{An_1}\frac{1-\cos\alpha}{\sin\alpha}\frac{\partial F}{\partial \alpha} - \frac{3Mm}{An_1}\cot\beta\frac{\partial F}{\partial \beta} \ldots(11),$$

$$\frac{d\phi_1}{dt} = -\frac{3Mm}{An_1}\frac{C-A}{C}\cot\alpha\cos\beta\frac{\partial F}{\partial \alpha} + \frac{3Mm}{n_1}\operatorname{cosec}\beta\left(\frac{\sin^2\beta}{A}+\frac{\cos^2\beta}{C}\right) - \frac{3Mm}{An_1}\frac{C-A}{C}\cos\beta\{\int n_1 dt+\varepsilon_1-N-\tfrac{1}{2}\pi\}\frac{\partial F}{\partial \varepsilon_1} \ldots(12).$$

The equations (3) and (9) give

$$\frac{na^2\cos\phi\cos i}{\mu}+\frac{A}{Mm}n_1\cos\alpha = \text{const},$$

or
$$An_1\cos\alpha+\frac{Mm}{\mu}na^2\cos\phi\cos i = h_0.$$

Taking the plane of reference to be the Invariable Plane, it follows that $An_1\sin\alpha-\frac{Mm}{\mu}na^2\cos\phi\sin i=0$, so that

$$\frac{dN}{dt}=\frac{d\Omega}{dt}.$$

Therefore $N=\Omega$ always.

§ 15. *The Secular Inequalities.*

The secular part of the disturbing function is $3F_0$, where

$$F_0 = \Sigma_1^\infty \frac{A_p}{a^{2p+1}}P_{2p}(\cos\beta)\sec^{2p}\phi\,[\mathfrak{P}_0^{2p}(\cos j)\,\mathfrak{P}_0^{2p-1}\sec\phi + 2\Sigma B_q\sin^{2q}j\,\mathfrak{P}_{2q}^{2p}(\cos j)\tan^{2q}\phi\,\mathfrak{P}_{2q}^{2p-1}(\sec\phi)\cos 2q(\varpi-N)].$$

B_q is simply a numerical constant.

A_p depends on the distribution of mass in the spheroid,

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and is of order $2p$ in the mean radius of the spheroid, and p in the ellipticity.

F_0 does not contain ε or ε_1.

Therefore $$\frac{da}{dt} = 0, \text{ and } \frac{dn_1}{dt} = 0.$$

Thus the satellite's mean distance has no secular inequality, neither has the inclination of the planet's axis to its axis of resultant angular momentum.

These results both hold, if there be other disturbing forces, for ε only occurs with nt and ε_1 with $n_1 t$. As regards the mean distance, the result is well known. The second result may be stated—The instantaneous axis of a planet describes a cone about its axis figure, whose angle suffers no secular change. The assumptions made are that the planet is rigid, and that n_1/n is not a whole number. That is, there must be no force upon it whose period is an exact multiple of the period of the motion of its axis in space.

§ 16. To find the secular variations of α, i, N, ϖ, ϕ, we have the equations

$$\sin\phi \frac{d\phi}{dt} = -3na \frac{\partial F_0}{d\varpi},$$

$$\frac{di}{dt} = 3na \sec\phi \cot j \frac{\partial F_0}{\partial \varpi},$$

$$\frac{da}{dt} = 3 \frac{Mm}{An_1} \operatorname{cosec} j \frac{\partial F_0}{\partial \varpi},$$

$$\frac{dN}{dt} = 3 \frac{Mm}{An_1} \operatorname{cosec} \alpha \frac{\partial F_0}{\partial j},$$

$$\frac{d\varpi}{dt} = 3na \left\{ \operatorname{cosec}\phi \frac{\partial F_0}{\partial \phi} + \tan \tfrac{1}{2} i \sec\phi \frac{\partial F_0}{\partial j} \right\}.$$

Therefore $$\frac{dj}{dt} = 3 \left(na \sec\phi \cot j + \frac{Mm}{An_1} \operatorname{cosec} j \right) \frac{\partial F_0}{\partial \varpi},$$

and $$\frac{d(\varpi - N)}{dt} = 3 \left\{ na \operatorname{cosec}\phi \frac{\partial F_0}{\partial \phi} + \left(na \tan \tfrac{1}{2} i \sec\phi - \frac{Mm}{Ad_1} \operatorname{cosec} \alpha \right) \frac{\partial F_0}{\partial j} \right\}.$$

Put $\varpi - N = v$.

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It is easily shewn that

$$p = f_0 + \frac{f_0'' f_0 \phi_1^2 + 4 f_0' f_0^2 f_1 \phi_1 - 4 f_0^2 (f_1^2 - f_1' \phi_1')}{16 f_0^3} + \&c.,$$

where f_0 is written for $f_0(x_0)$, &c., x_0 being the initial value of x, and f_0' for $\frac{d}{dt} f_0(x)$, when t is put $= 0$ after the differentiation.

Thus p is given correctly to cubes of the small quantities.

The values of a_1, b_1, &c. are easily found, and we obtain

$$\left.\begin{aligned} x &= a_0 - \frac{\phi_1}{2 f_0} \cos 2(v_0 + pt) \\ &\quad + \frac{\phi_1 (2 f_1 f_0 - \phi_1 f_0' - \phi_1' f_0) + 4 \phi_2 f_0^2}{16 f_0^3} \cos 4(v_0 + pt) + \dots \\ v &= v_0 + pt - \frac{f_0' \phi_0}{4 f_0^2} \sin 2(v_0 + pt) + \dots \end{aligned}\right\}.$$

Substitution in the equations of § 16 shews that

$$e,\ i,\ \alpha,\ \frac{d\varpi}{dt},\ \frac{dN}{dt} \text{ are all of the forms}$$

$$A_0 + A_1 \cos 2(v_0 + pt) + A_2 \cos 4(v_0 + pt) + \&c.,$$

where A_1, A_2, A_3, &c. are of the 1st, 2nd, 3rd, &c., orders in the ellipticity of the planet and the square of the ratio of its mean radius to the mean distance of the satellite. Therefore to the first order of the masses, we find that the secular inequalities of a spheroidal planet and its satellite are such that the mean distance and the inclination of the planet's axis of figure to the line of its resultant axis of momentum remain constant: the eccentricity and inclination of the satellite and of the planet's axis of momentum to the invariable plane of the system, the velocity of the apse and node remain nearly constant: but have periodic changes: the periods being in each case submultiples of the time taken by the apse line from the invariable plane to it again.

§ 18. There are two interesting special cases.

When i, and therefore α and j, vanish, then F_0 is independent of $\varpi - N$, and we have

$$e \text{ and } \frac{d\varpi}{dt} \text{ constant.}$$

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Neglecting the eccentricity of the satellite's orbit

$$\frac{\alpha}{i} = \frac{Ma^2}{C} \frac{m}{m+M} \frac{n}{\omega}.$$

If the planet be the earth and the satellite the moon,

$$\frac{m}{M+m} = \frac{1}{82{\cdot}315},$$

$$\frac{n}{\omega} = \frac{1}{27{\cdot}322},$$

Moon's mean parallax $= 57'1''8$.

Therefore $\quad \frac{\alpha}{i} = 4{\cdot}0153.$

Also $\quad \frac{C-A}{C} = {\cdot}003273.$

Thus, the node advances through four right angles in about 6800 years, and the apse regredes through four right angles in about 8500 years.

§ 19. *The Periodic Inequalities.*

These do not possess much interest in the general case: we shall only consider the case where β is small. There is a little difficulty, owing to the presence of β in the denominator on the right-hand side of the equation for $\frac{d\varpi_1}{dt}$.

The equations are integrated just as in the ordinary planetary theory, where the inclinations are small.

The result will be applied to find the moon's librations; considering it as a spheroidal planet, having the earth as a spherical satellite. It is on this account that the terms in $(n-n_1)t$ are retained in the equations, as they will rise in importance, on integration, when $n = n_1$.

Neglecting β^2, we find

$$F = \frac{A_1}{r^3}\{-\tfrac{1}{2}P_2(\cos j) - \tfrac{3}{4}\sin^2 j \cos 2(\theta - N)$$

$$+ \tfrac{3}{4}\beta[\sin 2j \sin(\lambda - N) - \sin 2j \sin(\lambda - N)\cos 2(\theta - N)$$

$$+ 2\sin j \cos(\lambda - N)\sin 2(\theta - N)]\}.$$

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Now $\frac{a^3}{r^3} = 1 + 3e\cos(nt + \varepsilon - \varpi) + \&c.,$

$$\frac{a^3\cos 2(\theta - \varpi)}{r^3} = \cos 2(nt + \varepsilon - \varpi) + \tfrac{1}{2}e[\cos(nt + \varepsilon - \varpi) + 5\cos 3(nt + \varepsilon - \varpi)] + \&c.,$$

$$\frac{a^3\sin 2(\theta - \varpi)}{r^3} = \sin 2(nt + \varepsilon - \varpi) + \tfrac{1}{2}e[\sin(nt + \varepsilon - \varpi) + 5\sin 3(nt + \varepsilon - \varpi)] + \&c.$$

On substitution, we obtain

$$\begin{aligned} F = \frac{A_1}{a^3}\{&-\tfrac{1}{2}P_2(\cos j) - \tfrac{3}{4}\sin^2 j\cos 2(nt + \varepsilon - N) \\ &+ \tfrac{3}{4}\beta\sin 2j\sin(n_1 t + \varepsilon_1 - N) \\ &+ \tfrac{3}{4}\beta\sin j(1 - \cos j)\sin(2nt + 2\varepsilon + n_1 t - \varepsilon_1 - 3N) \\ &+ \tfrac{3}{4}\beta\sin j(1 + \cos j)\sin(2nt + 2\varepsilon - n_1 t - \varepsilon_1 - N) \\ &- \tfrac{3}{8}\beta e\sin 2j\sin(nt + \varepsilon - n_1 t - \varepsilon_1 - \varpi + N) \\ &- \tfrac{3}{8}\beta e\sin j(1 + \cos j)\sin(nt + \varepsilon - n_1 t + \varepsilon_1 + \varpi - N)\}. \end{aligned}$$

Consider any one of these terms, for instance

$$\tfrac{3}{4}\frac{A_1}{a^3}\beta\sin j(1 - \cos j)\sin(2nt + 2\varepsilon + n_1 t + \varepsilon_1 - 3N).$$

This will give rise to a term having β as a factor in $\frac{da}{dt}, \frac{de}{dt}, \frac{di}{dt}, \frac{dN}{dt}$, and these need not be considered.

The equations for $\frac{d\beta}{dt}$ and $\frac{d\varepsilon}{dt}$ are

$$C\omega\beta\frac{d\beta}{dt} = 3Mm\frac{\partial F}{\partial\varepsilon_1},$$

and

$$C\omega\beta\frac{d\varepsilon_1}{dt} = -3Mm\frac{\partial F}{\partial\beta}.$$

Therefore

$$\frac{d\beta}{dt} = \frac{3Mm}{C\omega}\tfrac{3}{4}\frac{A_1}{a^3}\sin j(1 - \cos j)\cos(2nt + 2\varepsilon + n_1 t + \varepsilon_1 - 3N),$$

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and

$$\beta \frac{d\varepsilon_1}{dt} = -\frac{3Mm}{C\varpi} \frac{3}{4} \frac{A_1}{a^3} \sin j (1-\cos j) \sin (2nt + 2\varepsilon + n_1 t + \varepsilon_1 - 3N).$$

For brevity, let these be written

$$\left.\begin{aligned} \frac{d\beta}{dt} &= \quad k \cos (K + \varepsilon_1) \\ \beta \frac{d\varepsilon_1}{dt} &= -k \sin (K + \varepsilon_1) \end{aligned}\right\}.$$

Therefore

$$\left.\begin{aligned} \frac{d}{dt} (\beta \sin \varepsilon_1) &= -k \sin K \\ \frac{d}{dt} (\beta \cos \varepsilon_1) &= \quad k \cos K \end{aligned}\right\}.$$

Let the mean values of $\frac{d\varpi}{dt}$ and $\frac{dN}{dt}$ (found in § 17), be c_1 and c_2.

Therefore $\beta \sin \varepsilon_1 = B_1 + k \frac{\cos K}{2n + n_1 - 3c_2}$

and $\beta \cos \varepsilon_1 = B_2 + k \frac{\sin K}{2n + n_1 - 3c_2}$

§ 20. When β is small, nothing is gained by finding the motion of the planet's axis of figure, relative to its line of resultant angular momemtum, as these lines nearly coincide. We shall therefore find the motion of the pole relative to the invariable plane of the system.

Let $\theta = ZP$, and $\psi = OM$.

The angle $RAP = n_1 t + \varepsilon_1 - N - \frac{1}{2}\pi$.

Therefore $\theta = \alpha + \beta \cos (n_1 t + \varepsilon_1 - N - \frac{1}{2}\pi)$,

and $\psi = N + \frac{1}{2}\pi + \beta \operatorname{cosec} \alpha \sin (n_1 t + \varepsilon_1 - N - \frac{1}{2}\pi)$.

The equations of § 19 give

$$\beta \sin (n_1 t + \varepsilon_1 - N - \tfrac{1}{2}\pi)$$
$$= B_1 \sin (n_1 t - N) - B_2 \cos (n_1 t - N) - \frac{k \sin 2 (nt + \varepsilon - N)}{2n + n_1 - 3c_2},$$

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[illegible] motion [illegible]

and [illegible]

$$= \text{[illegible]} - \frac{\cos^2 \quad \pi - \varepsilon - N}{2n - n_1 - c_2}.$$

[illegible] the complementary function, we find

$$\text{[illegible]} = \frac{dn}{\cos} \frac{1}{i^2} - \sin \quad 1 - \cos$$

$$\frac{\sin^2 \quad \pi - \varepsilon - N}{2n - n_1 - c_2} - \sin \quad 1 - \cos \quad \frac{\sin^2 \quad \pi - \varepsilon - N}{2n - n_1 - c_2}$$

$$\frac{\sin^2 \quad \pi - \varepsilon - \varpi}{n - n_1 - c_1 - c_2} - \frac{1}{2}e \sin \quad 1 - \cos \quad \frac{\sin \quad \pi - \varepsilon - \varpi - N}{n - n_1 - c_1 - c_2}.$$

and

$$\text{[illegible]} = \frac{dn}{\cos} \frac{1}{i^2} \cdot \frac{\sin 2}{n_1 - c_2} - \sin \quad 1 - \cos$$

$$\frac{\cos^2 \quad \pi - \varepsilon - N}{2n - n_1 - c_2} + \sin \quad 1 - \cos \quad \frac{\cos^2 \quad \pi - \varepsilon - N}{2n - n_1 - c_2}$$

$$\frac{\cos \quad \pi - \varepsilon + \varpi}{n - n_1 - c_1 - c_2} - \frac{1}{2}e \sin \quad 1 + \cos \quad \frac{\cos \quad \pi - \varepsilon + \varpi - N}{n - n_1 - c_1 - c_2}.$$

Again, $$\frac{dN}{dt} = \frac{3}{\cos} \frac{Mm}{a^2} \frac{1}{\sin \alpha} \frac{\partial F}{\partial \alpha}$$

$$= \frac{3}{\cos} \frac{Mm}{a^2} \frac{A_1}{\sin \alpha} \sin j \cos j \{1 - \cos 2 (nt + \varepsilon - N)\}.$$

Therefore

$$N = N_0 + n_1 t - \left\{ \frac{\sin 2 (nt + \varepsilon - N)}{2n - 2c_2} - \frac{1}{2}e \frac{\sin (nt + \varepsilon + \varpi - 2N))}{n + c_1 - 2c_1} \right\},$$

where n_1 is the mean motion of the node,

and $$= \frac{3}{\cos} \frac{Mm}{a^2} \frac{A_1}{\sin \alpha} \sin j \cos j.$$

$$\text{[illegible]} \frac{di}{dt} = B \frac{Mm}{\cos} \left\{ \frac{1}{\sin j} \left(\frac{\partial F}{\partial i} + \frac{\partial F}{\partial \varpi} \right) - \frac{\sin i}{\sin \alpha \sin j} \frac{\partial F}{\partial \varepsilon_1} \right\}$$

$$= \frac{Mm}{\cos} \frac{A_1}{a^2} \sin j \sin 2 (nt + \varepsilon - N).$$

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Thus $\alpha = \alpha_0 - C_2 \frac{\sin\alpha}{\cos j} \frac{\cos 2(nt+\varepsilon-N)}{2(n-c_2)}$.

Thus, the position of the planet's pole is given by

$$\theta = \theta_0 - \frac{\sin\alpha}{\cos j} c_2 \left\{ \cos 2(nt+\varepsilon-N) \left[\frac{1}{2(n-c_2)} - \frac{1-\cos j}{2(2n+n_1-3c_2)} - \frac{1+\cos j}{2(2n-n_1-c_2)} \right] + \tfrac{3}{2} e \cos j \frac{\cos(nt+\varepsilon-\varpi)}{n-n_1-c_1+c_2} + \tfrac{1}{2} e (1+\cos j) \frac{\cos(nt+\varepsilon+\varpi-2N)}{n-n_1+c_1-c_2} \right\},$$

and

$$\psi = \psi_0 + c_2 \left\{ t - \sin 2(nt+\varepsilon-N) \left[\frac{1}{2n-2c_2} + \frac{\sin\alpha(1-\cos j)}{2\cos j} \frac{1}{2n+n_1-3c_2} - \frac{\sin\alpha(1+\cos j)}{2\cos j} \frac{1}{2n-n_1-c_2} \right] - \tfrac{3}{2} e \sin\alpha \frac{\sin(nt+\varepsilon-\varpi)}{n-n_1-c_1+c_2} - \tfrac{1}{2} e \sin\alpha \frac{1+\cos j}{\cos j} \frac{\sin(nt+\varepsilon+\varpi-2N)}{n-n_1+c_1-c_2} \right\}.$$

In the case of the Moon α is very nearly equal to j,

$$n = n_1. \quad \text{Also } j = 6^\circ.44'.$$

Putting $\cos j = 1$, we find for the moon's librations

$$\theta = \theta_0 + \tfrac{3}{2} \sin j \frac{c_2}{n-c_2} \cos 2(nt+\varepsilon-N) - \tfrac{3}{2} e \sin j \frac{c_2}{c_2-c_1} \cos(nt+\varepsilon-\varpi) + \tfrac{1}{2} e \sin j \frac{c_2}{c_2-c_1} \cos(nt+\varepsilon+\varpi-2N),$$

and

$$\psi \sin j = \tfrac{3}{2} \sin j \frac{c_2}{n-c_2} \sin 2(nt+\varepsilon-N) - \tfrac{3}{2} e \sin j \frac{c_2}{c_2-c_1} \sin(nt+\varepsilon-\varpi) + \tfrac{1}{2} e \sin j \frac{c_2}{c_2-c_1} \sin(nt+\varepsilon+\varpi-2N).$$

The moon's actual librations consist of terms of the form found above: but the coefficients of the terms when numbers are substituted are entirely different, owing to the disturbing

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and $\beta\cos(n_1 t + \varepsilon_1 - N - \frac{1}{2}\pi)$

$$= B_1\cos(n_1 t - N) + B_2\sin(n_1 t - N) + \frac{k\cos 2(nt+\varepsilon-N)}{2n+n_1-3c_2}.$$

Omitting the complementary function, we find

$$\beta\sin(n_1 t+\varepsilon_1-N-\tfrac{1}{2}\pi) = \tfrac{9}{4}\frac{Mm}{C\omega}\frac{A_1}{a^3}\left\{-\sin j(1-\cos j)\right.$$

$$\times\frac{\sin 2(nt+\varepsilon-N)}{2n+n_1-3c_2} + \sin j(1+\cos j)\frac{\sin 2(nt+\varepsilon-N)}{2n-n_1-c_2}$$

$$\left.-\tfrac{3}{2}e\sin 2j\frac{\sin(nt+\varepsilon-\varpi)}{n-n_1-c_1+c_2} - \tfrac{1}{2}e\sin j(1+\cos j)\frac{\sin(nt+\varepsilon+\varpi-2N)}{n-n_1+c_1-c_2}\right\},$$

and

$$\beta\cos(n_1 t+\varepsilon_1-N-\tfrac{1}{2}\pi) = \tfrac{9}{4}\frac{Mm}{C\omega}\frac{A_1}{a^3}\left\{\frac{\sin 2j}{n_1-c_2}+\sin j(1-\cos j)\right.$$

$$\times\frac{\cos 2(nt+\varepsilon-N)}{2n+n_1-3c_2} + \sin j(1+\cos j)\frac{\cos 2(nt+\varepsilon-N)}{2n-n_1-c_2}$$

$$\left.-\tfrac{3}{2}e\sin 2j\frac{\cos(nt+\varepsilon-\varpi)}{n-n_1-c_1+c_2} - \tfrac{1}{2}e\sin j(1+\cos j)\frac{\cos(nt+\varepsilon+\varpi-2N)}{n-n_1+c_1-c_2}\right\}.$$

Again, $\dfrac{dN}{dt} = \dfrac{3Mm}{C\omega}\dfrac{1}{\sin\alpha}\dfrac{\partial F}{\partial\alpha}$

$$= \tfrac{9}{2}\frac{Mm}{C\omega}\frac{A_1}{a^3}\frac{\sin j\cos j}{\sin\alpha}\{1-\cos 2(nt+\varepsilon-N)\}.$$

Therefore

$$N = N_0 + c\left\{t - \frac{\sin 2(nt+\varepsilon-N)}{2n-2c_2} - \tfrac{1}{2}e\frac{\sin(nt++\varpi-2N)}{n+e_1-2c_1}\right\},$$

where c_2 is the mean motion of the node,

and $$= \tfrac{9}{2}\frac{Mm}{C\omega}\frac{A_1}{a^3}\frac{\sin j\cos j}{\sin\alpha}.$$

Also $$\frac{d\alpha}{dt} = 3\frac{Mm}{C\omega}\left\{\frac{1}{\sin j}\left(\frac{\partial F}{\partial\varepsilon}+\frac{\partial F}{\partial\varpi}\right) - \frac{\sin i}{\sin\alpha\sin j}\frac{\partial F}{\partial\varepsilon_1}\right\}$$

$$= \tfrac{9}{2}\frac{Mm}{C\omega}\frac{A_1}{a^3}\sin j\sin 2(nt+\varepsilon-N).$$

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Thus $\alpha = \alpha_0 - C_2 \frac{\sin\alpha}{\cos j} \frac{\cos 2\,(nt + \varepsilon - N)}{2\,(n - c_2)}$.

Thus, the position of the planet's pole is given by

$$\theta = \theta_0 - \frac{\sin\alpha}{\cos j} c_2 \left\{ \cos 2\,(nt + \varepsilon - N) \left[\frac{1}{2\,(n - c_2)} - \frac{1 - \cos j}{2\,(2n + n_1 - 3c_2)} - \frac{1 + \cos j}{2\,(2n - n_1 - c_2)} \right] \right.$$

$$\left. + \tfrac{3}{2} e \cos j \frac{\cos(nt + \varepsilon - \varpi)}{n - n_1 - c_1 + c_2} + \tfrac{1}{2} e\,(1 + \cos j) \frac{\cos(nt + \varepsilon + \varpi - 2N)}{n - n_1 + c_1 - c_2} \right\},$$

and

$$\psi = \psi_0 + c_2 \left\{ t - \sin 2\,(nt + \varepsilon - N) \left[\frac{1}{2n - 2c_2} + \frac{\sin\alpha\,(1 - \cos j)}{2\cos j} \frac{1}{2n + n_1 - 3c_2} - \frac{\sin\alpha\,(1 + \cos j)}{2\cos j} \frac{1}{2n - n_1 - c_2} \right] \right.$$

$$\left. - \tfrac{3}{2} e \sin\alpha \frac{\sin(nt + \varepsilon - \varpi)}{n - n_1 - c_1 + c_2} - \tfrac{1}{2} e \sin\alpha \frac{1 + \cos j}{\cos j} \frac{\sin(nt + \varepsilon + \varpi - 2N)}{n - n_1 + c_1 - c_2} \right\}.$$

In the case of the Moon α is very nearly equal to j,

$$n = n_1. \quad \text{Also } j = 6°.44'.$$

Putting $\cos j = 1$, we find for the moon's librations

$$\theta = \theta_0 + \tfrac{3}{2} \sin j \frac{c_2}{n - c_2} \cos 2\,(nt + \varepsilon - N)$$

$$- \tfrac{3}{2} e \sin j \frac{c_2}{c_2 - c_1} \cos(nt + \varepsilon - \varpi) + \tfrac{1}{2} e \sin j \frac{c_2}{c_2 - c_1} \cos(nt + \varepsilon + \varpi - 2N),$$

and

$$\psi \sin j = \tfrac{3}{2} \sin j \frac{c_2}{n - c_2} \sin 2\,(nt + \varepsilon - N)$$

$$- \tfrac{3}{2} e \sin j \frac{c_2}{c_2 - c_1} \sin(nt + \varepsilon - \varpi) + \tfrac{1}{2} e \sin j \frac{c_2}{c_2 - c_1} \sin(nt + \varepsilon + \varpi - 2N).$$

The moon's actual librations consist of terms of the form found above: but the coefficients of the terms when numbers are substituted are entirely different, owing to the disturbing

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action of the sun having been omitted, and a spheroidal instead of ellipsoidal form having been assumed for the moon.

§ 21. *The stability of the system.*

When only the terms of the first order in the disturbing force are considered, and if there is no exact ratio between n and n_1, we have shewn that a and β have no secular variations: and that e, i, α, $\frac{d\varpi}{dt}$, $\frac{dN}{dt}$ are nearly constant, but have long inequalities whose periods are submultiples of the period of $2(\varpi - N)$. All these quantities have, in addition, periodic inequalities.

A typical term of the disturbing function is

$$xA\cos\{i[\int n_1 dt + \varepsilon_1 - N] + j[\int n dt + \varepsilon - \varpi] + k[\varpi - N]\},$$

where A is a fraction of a, e, i, α, β: and x indicates that we are considering terms of the first order in the disturbing mass.

Let this term be written $xA\cos D$.

It will give rise to terms in the variations of a, e, i, &c., of the following forms:

$$\left.\begin{aligned} \frac{da}{dt} &= xF_1\sin D \\ \frac{de}{dt} &= xF_2\sin D \\ \frac{di}{dt} &= xF_3\sin D \\ \frac{d\alpha}{dt} &= xF_4\sin D \\ \frac{d\beta}{dt} &= xF_5\sin D \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{d\varpi}{dt} &= xG_1\cos D \\ \frac{dN}{dt} &= xG_2\cos D \\ \frac{d\varepsilon}{dt} &= xG_3\cos D \\ \frac{d\varepsilon_1}{dt} &= xG_4\cos D \end{aligned}\right\}.$$

Taking into account the secular inequalities as well, we find

$$a = a_0 - x\Sigma\frac{F_1}{in_1 + jn}\cos D,$$

$$e = e_0 - x\Sigma P_1\cos 2s(pt+v) - x\Sigma\frac{F_2}{in_1 + jn}\cos D,$$

$$i = i_0 - x\Sigma P_2\cos 2s(pt+v) - x\Sigma\frac{F_3}{in_1 + jn}\cos D,$$

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$$\alpha = \alpha_0 - x\Sigma P_s \cos 2s\,(pt+v) - x\Sigma \frac{F_s}{in_1+jn} \cos D,$$

$$\beta = \beta_0 - x\Sigma \frac{F_s}{in_1+jn} \cos D,$$

$$\varpi = \varpi_0 + c_1 t + x\Sigma Q_s \sin 2s\,(pt+v) + x\Sigma \frac{G_1}{in_1+jn} \sin D,$$

$$N = N_0 + c_2 t + x\Sigma Q_s \sin 2s\,(pt+v) + x\Sigma \frac{G_2}{in_1+jn} \sin D,$$

$$\varepsilon = \varepsilon_0 + c_3 t + x\Sigma Q_s \sin 2s\,(pt+v) + x\Sigma \frac{G_3}{in_1+jn} \sin D,$$

$$\varepsilon_1 = \varepsilon_{1,0} + c_4 t + x\Sigma Q_s \sin 2s\,(pt+v) + x\Sigma \frac{G_4}{in_1+jn} \sin D.$$

Or if we take into account the constant parts of $\frac{d\varpi}{dt}$, &c., in the integration of $\sin D$, $\cos D$, we must divide by

$$i(n_1+c_4-c_3)+j(n+c_3-c_1)+k(c_1-c_2) \text{ instead of } in_1+jn.$$

Write, for brevity,

$$a = a_0 + d_1 a_0, \quad \varpi = \varpi_0 + c_1 t + \delta_1 \varpi_0 + \delta_1 c_1 t.$$

Then $\delta_1 \int n dt = -\frac{3}{2} \frac{n_0}{a_0} f \delta_1 a_0 dt$

$$= +\frac{3}{a_0^2} \Sigma \frac{xF_1}{[i(n_1+c_4-c_3)^2 + \&c]^2} \sin D.$$

A similar result holds for

$$\delta_1 \int n_1 dt.$$

Substitute these disturbed values of a_1, &c., in the right-hand side of the equations

$$\left.\begin{aligned} \frac{da}{dt} &= x\Sigma F_1 \sin D \\ \text{and} \quad \frac{d\beta}{dt} &= x\Sigma F_2 \sin D \end{aligned}\right\}.$$

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Therefore $\frac{da}{dt}$ is of the form

$$\varkappa\Sigma F_1 \sin D + \varkappa^2 \Sigma K_1 \sin 2D + \Sigma\Sigma L_1 \sin (D \pm D') + \Sigma\Sigma M_1 \sin [D \pm 2s(pt+v)]\}.$$

Therefore, to the second order,

$$a = a_0 + \text{periodic terms.}$$

These periodic terms are of short period. The long inequality of period $\frac{2\pi}{p}$, which appears in e, i, &c. and arose from the integration of the secular terms of the disturbing function, does not occur in a or β.

§ 22. These results do not hold when the ratio of n_1 to n is that of two integers.

Let j and j' be two positive integers, such that

$$j'/j = n/n_1.$$

Thus the term

$$\varkappa A \cos \{j' [\int n_1 dt + \varepsilon_1 - N] - j [\int n dt + \varepsilon - \varpi] + k [\varpi - N]\}$$

will give rise to secular variations in a and β.

Considering only these terms,

$$j' \frac{\partial F}{\partial \varepsilon} - j \frac{\partial F}{\partial \varepsilon_1} = 0.$$

Therefore $$\frac{j'}{2na^2} \delta a + \frac{jC\omega}{mM} \delta (\sec \beta) = 0,$$

or $$\delta n_1 = \tfrac{1}{8} j' \frac{mMa^2}{A(M+m)} \delta n.$$

The inequalities produced are generally of long period: that is, of a length similar to the periods of the node and apse: for when $j [\int n_1 dt + \varepsilon_1] = j [\int n dt + \varepsilon]$, the term of the disturbing function becomes

$$\varkappa A \cos \{j\varpi - j'N + k\varpi - kN + \alpha\}.$$

In the case where $j = j'$: it might happen that $k = -j$ and then the term would not be periodic.

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Referring to the expression for the disturbing function in § 12, these terms are contained in

$$\tfrac{3}{8}\frac{A_1}{a^3}\sin^2\beta\,(1+\cos^2 j)\cos 2\,(\lambda - N)\cos 2\,(\theta - N),$$

$$\tfrac{3}{4}\frac{A_1}{a^3}\sin^2\beta\cos j \qquad \sin 2\,(\lambda - N)\cos 2\,(\theta - N),$$

and thus the term producing this non-periodic change in a and β is

$$\tfrac{3}{16}\frac{A_1}{a^3}\sin^2\beta\,(1+\cos j)^2\cos 2\,\{\int n\,dt + \varepsilon - \int n_1 dt - \varepsilon_1\}.$$

This term is not of any practical interest, because β is very nearly zero in all actual cases: and if it were not zero the frictional forces in the body itself would have much greater effect in changing its value.

ON CHREE'S PROBLEM OF THE ROTATING ELASTIC ELLIPSOID.

By D. Edwardes, B.A.

THE problem appears as one of the applications made by Mr. Chree* of his general solution of the equations of Elasticity, and he has worked out the case of the spheroid to the final expressions for the displacements.

Perhaps this might be advantageously considered as a separate problem, and an independent investigation which should arrive at Mr. Chree's results without any actual process of integration be of service. The following analysis refers therefore principally to the case of a rotating spheroid, but a process is conducted so far as to be assured that there are nine linear equations to determine the nine constants in the case of the ellipsoid.

Let us consider then an oblate spheroid rotating about its axis with angular velocity ω. The symmetry of the con-

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* *Quarterly Journal*, Vols. XXII, XXIII.

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ditions allows us to write down the equations in cylindrical coordinates, as follows:

$$\left.\begin{aligned}(m+n)\frac{d\Delta}{d\varpi}+2n\frac{d\theta}{dz}+\rho\omega^2\varpi=0\\(m+n)\frac{d\Delta}{dz}-\frac{2n}{\varpi}\frac{d}{d\varpi}(\varpi\theta)=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

where the dilatation and the rotation are expressed in terms of the displacements by

$$\Delta=\frac{du}{d\varpi}+\frac{dw}{dz}+\frac{u}{\varpi}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$2\theta=\frac{du}{dz}-\frac{dw}{d\varpi}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The third term in the first of equations (1) being linear in the variable, we seek to obtain expressions for the displacements which shall be rational, integral functions of ϖ, z, of a degree not higher than the third, and which, subject to certain conditions as to symmetry, shall satisfy these equations and the boundary conditions.

Let U_s, W_s be rational, integral, homogeneous functions of ϖ, z; β_s a numerical constant, and consider the partial displacements—

$$\left.\begin{aligned}u_s&=\varpi\left\{\frac{dU_s}{d\varpi}+\frac{dW_s}{dz}+\frac{U_s}{\varpi}\right\}+\beta_s U_s\\w_s&=z\left\{\frac{dU_s}{d\varpi}+\frac{dW_s}{dz}+\frac{U_s}{\varpi}\right\}+\beta_s W_s\end{aligned}\right\}\quad\ldots\ldots(4).$$

Then simplifying by Euler's theorem, the suffix indicating the degree, we have

$$\Delta_s=(s+2+\beta_s)\left(\frac{dU_s}{d\varpi}+\frac{dW_s}{dz}+\frac{U_s}{\varpi}\right)\quad\ldots\ldots\ldots(5).$$

Let us assume now that U_s, W_s satisfy respectively the equations

$$\frac{d^2U_s}{d\varpi^2}+\frac{d^2U_s}{dz^2}+\frac{1}{\varpi}\frac{dU_s}{d\varpi}-\frac{U_s}{\varpi^2}=0\quad\ldots\ldots\ldots\ldots(6),$$

$$\frac{d^2W_s}{d\varpi^2}+\frac{d^2W_s}{dz^2}+\frac{1}{\varpi}\frac{dW_s}{d\varpi}=0\quad\ldots\ldots\ldots\ldots(7).$$

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We find then

$$2\theta_s = (s+\beta_s)\left(\frac{dU_s}{dz} - \frac{dW_s}{d\varpi}\right) \quad \ldots\ldots\ldots\ldots(8).$$

If $\varpi U_s = \psi_{s+1}$, and $\varpi \dfrac{dW_s}{d\varpi} = \psi_s$, we see that ψ_s and ψ_{s+1} are integral homogeneous solutions, of degrees s and $s+1$ respectively, of

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0 \ldots\ldots\ldots\ldots(9),$$

Stokes's Current-Function Equation.

Omitting for the present the acceleration, and substituting in equations (1), the left hand members become respectively

$$M_s\left(\frac{d^2W_s}{d\varpi dz} - \frac{d^2U_s}{dz_s}\right), \text{ and } M_s\left(\frac{d^2U_s}{d\varpi dz} + \frac{d^2W_s}{dz^2} + \frac{1}{\varpi}\frac{dU_s}{dz}\right),$$

where $$M_s = (s+2)m + 2n + m\beta_s,$$

and are therefore satisfied provided

$$\beta_s = -\{(s+2)m + 2n\}/m.$$

Hence, if there are no bodily forces, and the displacements are functions of ϖ and z alone, a solution is $u = \Sigma u_s$, $w = \Sigma w_s$, in combination with (6) and (7).

Mr. Sampson in his memoir "On Stokes's Current Function"* has shown how to construct solutions of equation (9) which are rational, integral, homogeneous functions of ϖ and z, and he has given a table of the first ten functions. We have, as far as the fourth degree,

$$\psi_1 = z,\ \psi_2 = \varpi^2,\ \psi_3 = \varpi^2 z,\ \psi_4 = \varpi^2(4z^2 - \varpi^2).$$

In the problem before us it is clear that u must contain ϖ as a factor, and w must contain z. Hence ψ_{n+1} must contain ϖ^2 as a factor, and ψ_n must contain ϖz. Remembering (7) we assume therefore

$$u = u_3 + u_1,\ w = w_3 + w_1,$$

where u_1, w_1, u_3, w_3 are determined by (4), and

$$U_3 = A'\varpi(4z^2 - \varpi^2),\quad W_3 = A''z(3\varpi^2 - 2z^2),$$
$$U_1 = B'\varpi,\qquad\qquad W_1 = B''z,$$

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* *Phil. Trans.* A. 1891.

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A', B', A'', B'' being arbitrary constants, also

$$\beta_1 = -(3m+2n)/m,\ \beta_2 = -(5m+2n)/m.$$

Consequently

$$\left.\begin{aligned} u &= \varpi\left\{\varpi^2\left(3A''+\frac{m+2n}{m}A'\right)-2z^2\left(3A''+\frac{2(3m+2n)}{m}A'\right)\right. \\ &\qquad \left. +B'-\frac{m+2n}{m}B'\right\} \\ w &= z\left\{-2\varpi^2\left(2A'+\frac{3(2m+n)}{m}A''\right)+4z^2\left(2A'+\frac{m+n}{m}A''\right)\right. \\ &\qquad \left. +2B'-\frac{2(m+n)}{m}B'\right\} \end{aligned}\right\} \ldots(10),$$

$$\Delta = -\frac{2n}{m}\left\{\varpi^2(3A''-4A')+2z^2(4A'-3A'')+2B'+B''\right\} \ldots(11).$$

Again, equations (1) are satisfied by

$$\left.\begin{aligned} u &= -\tfrac{1}{8}\frac{\rho}{m+n}\omega^2\varpi^2 \\ w &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(12),$$

giving

$$\Delta = -\tfrac{1}{2}\frac{l}{m+n}\omega^2\varpi^2 \ldots\ldots\ldots\ldots\ldots(13).$$

Let the equation of the bounding surface be $\varpi^2/a^2+z^2/c^2=1$. The cylindrical coordinates being ϖ, θ, z and everything being independent of θ, with no displacement in longitude, the stresses S and U (referred to this system) vanish, and the direction cosines of the normal to the boundary (referred to this system) are $p\varpi/a^2$, 0, pz/c^2. At the boundary we have therefore

$$\left.\begin{aligned} \frac{\varpi}{a^2}\left\{(m-n)\Delta+2n\frac{du}{d\varpi}\right\}+\frac{nz}{c^2}\left\{\frac{du}{dz}+\frac{dw}{d\varpi}\right\} &= 0 \\ \frac{n\varpi}{a^2}\left\{\frac{du}{dz}+\frac{dw}{d\varpi}\right\}+\frac{z}{c^2}\left\{(m-n)\Delta+2n\frac{dw}{dz}\right\} &= 0 \end{aligned}\right\} \ldots\ldots(14).$$

Calculating separately the strains resulting from (10) and (12), adding the results and substituting in (14), dividing out ϖ from the first equation and z from the second, multiplying terms which involve only constants by

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$\varpi^2/a^2 + z^2/c^2$, and equating to zero the coefficients of ϖ^2 and z^2, we have

$$\left.\begin{aligned}
&(7m+2n)A' + 3(2m+n)A'' - 3ma^{-2}B' + na^{-2}B' \\
&\qquad = \tfrac{1}{2}H(2m+n)\omega \\
&4\{5ma^{-2} + 2(2m+n)c^{-2}\}A' + 6\{na^{-2} + (3m+n)c^{-2}\}A'' \\
&\qquad + 3ma^{-2}c^{-2}B' - na^{-2}c^{-2}B' = 0 \\
&8(2m+n)A' + 6(3m+n)A'' + 2nB'c^{-2} \\
&\qquad - (3m+n)B'c^{-2} = 0 \\
&4\{2(2m+n)a^{-2} + nc^{-2}\}A' + 3\{2(3m+n)a^{-2} \\
&\qquad + 5(m+n)c^{-2}\}A'' \\
&- 2na^{-2}c^{-2}B' + (3m+n)a^{-2}c^{-2}B' = -H(m-n)c^{-2}\omega^2
\end{aligned}\right\} \ldots(15),$$

where $$H = \tfrac{1}{2}\rho\frac{m}{n(m+n)}.$$

If $$I = \begin{vmatrix} 20ma^{-2} + (23m+10n)c^{-2}, & 2na^{-2} + (8m+3n)c^{-2} \\ 16(2m+n)a^{-2} + 4nc^{-2}, & 4(3m+n)a^{-2} + (5m+n)c^{-2} \end{vmatrix},$$

then writing $c^2 = a^2(1-\varepsilon)$, we have

$$I = c^{-4}(5m+2n)\{5(19m-5n) - 40\varepsilon(3m-n)\} + 16\varepsilon^2(3m-n)\},$$

and the above equations can be handled so as to give without much trouble

$$I\left\{3A'' + \frac{2(3m+2n)}{m}A'\right\} = \frac{H\omega^2}{c^4}\frac{5m+2n}{m} \times \left\{15m^2 + 26mn - 5n^2 - 4m(m+5n)\varepsilon\right\},$$

$$I\left\{2A' + \frac{m+n}{m}A''\right\} = \frac{H\omega^2}{c^4}(5m+2n)\left\{5m - 3n - \frac{4(m^2-n^2)}{3m}\varepsilon\right\},$$

$$I\left\{2A' + \frac{3(2m+n)}{m}A''\right\} = \frac{4H\omega^2}{c^4}\frac{5m+2n}{m} \times \{-m(5m+n) + \varepsilon(4m^2 + mn + n^2)\},$$

$$I\left\{3A'' + \frac{m+2n}{m}A' - \tfrac{1}{8}\frac{\rho}{m+n}\omega^2\right\} = \frac{H\omega^2}{c^4}(5m+2n) \times \left\{-\frac{2(5m^2+16mn-5n^2)}{m} + \frac{4}{m}(2m^2+13mn-5n^2)\varepsilon - \frac{8n}{m}(3m-n)\varepsilon^2\right\} \ldots\ldots\ldots\ldots(16),$$

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$$mc^4 I\left(B' - \frac{m+2n}{m} B\right) = a^2 H\omega^2 \left\{\frac{16m(5m^2+10mn-3n^2)}{3m-n}\right.$$

$$\left. - 2\varepsilon(15m^2+46mn-n^2) + 8\varepsilon^2 m(m+5n)\right\}(5m+2n),$$

$$2mc^4 I = a^2 H\omega^2 (5m+2n)\left\{\frac{-4(40m^3-15m^2n-18mn^2+5n^3)}{8m-n}\right.$$

$$\left. + 4\varepsilon(15m^2-8n^2-mn) - 16(m^2-n^2)\varepsilon^2\right\}.$$

Adding the expressions in (10) and (12) and substituting we obtain Mr. Chree's results, observing that $c^4 I = 2(5m+2n)D$. Then Δ and θ are given by (2) and (3).

Considering now the case of an ellipsoid, we write down the equations

$$\left.\begin{aligned} m\frac{d\Delta}{dx} + n\nabla^2 u + \rho\omega^2 x &= 0 \\ m\frac{d\Delta}{dy} + n\nabla^2 v + \rho\omega^2 y &= 0 \\ m\frac{d\Delta}{dz} + n\nabla^2 w &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(17),$$

where $$\Delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

Let U_s, V_s, W_s be solid harmonics of positive degree s, and omitting for the present bodily forces, consider the partial displacements,

$$\left.\begin{aligned} u_s &= x\left(\frac{dU_s}{dx} + \frac{dV_s}{dy} + \frac{dW_s}{dz}\right) + \beta_s U_s \\ v_s &= y\left(\frac{dU_s}{dx} + \frac{dV_s}{dy} + \frac{dW_s}{dz}\right) + \beta_s V_s \\ w_s &= z\left(\frac{dU_s}{dx} + \frac{dV_s}{dy} + \frac{dW_s}{dz}\right) + \beta_s W_s \end{aligned}\right\} \quad \ldots\ldots\ldots(18).$$

Then $$\Delta_s = (s+2+\beta_s)\left(\frac{dU_s}{dx} + \frac{dV_s}{dy} + \frac{dW_s}{dz}\right),$$

and $$\nabla^2 u_s = 2\frac{d}{dx}\left(\frac{dU_s}{dx} + \frac{dV_s}{dy} + \frac{dW_s}{dz}\right),$$

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so that the equations of equilibrium under no bodily forces are satisfied provided $\beta_s = -\{(s+2)\,m + 2n\}/m$, and we can write

$$u = \Sigma u_s,\quad v = \Sigma v_s,\quad w = \Sigma w_s \ldots\ldots\ldots\ldots\ldots (19).$$

If we put $U_s = x\{3(B_1y^2 + C_1z^2) - (B_1 + C_1)x^2\}$,

$$V_s = y\{3(C_2z^2 + A_2x^2) - (C_2 + A_2)y^2\},$$

$$W_s = z\{3(A_3x^2 + B_3y^2) - (A_3 + B_3)z^2\},$$

$U_1 = D_1x$, $V_1 = D_2y$, $W_1 = D_3z$, then $u = u_s + u_1$, $v = v_s + v_1$, $w_s + w_1$ will be a solution, where

$$u_s = x\left[x^2\left\{3(A_2 + A_3) + \frac{2(m+n)}{m}(B_1 + C_1)\right\}\right.$$

$$+ 3y^2\left\{B_3 - \frac{2(2m+n)}{m}B_1 - C_2 - A_3\right\}$$

$$\left. + 3z^2\left\{-\frac{2(2m+n)}{m}C_1 + C_2 - A_3 - B_3\right\}\right]$$

with similar expressions for v_s and w_s, and

$$u_1 = x\left\{-\frac{2(m+n)}{m}D_1 + D_2 + D_3\right\}$$

with similar expressions for v_1 and w_1. Also, a particular solution of (17) is

$$u = -\tfrac{1}{8}\frac{\rho\omega^2}{m+n}x(x^2 + y^2),$$

$$v = -\tfrac{1}{8}\frac{\rho\omega^2}{m+n}y(x^2 + y^2),$$

$$w = 0,$$

giving $$\Delta = -\tfrac{1}{2}\frac{\rho\omega^2}{m+n}(x^2 + y^2).$$

At the boundary we have three equations such as

$$\frac{x}{a^2}\left\{(m-n)\Delta + 2n\frac{du}{dx}\right\} + \frac{ny}{b^2}\left(\frac{dv}{dx} + \frac{du}{dy}\right) + \frac{nz}{c^2}\left(\frac{dw}{dx} + \frac{du}{dz}\right) = 0,$$

where $u = u_s + u_1 - \tfrac{1}{8}\dfrac{\rho\omega^2}{m+n}x(x^2 + y^2)$, $v =$ &c.,

$w = w_s + w_1$,

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NOTE ON ELLIPTIC CYLINDRICAL VORTICES.

By A. E. H. Love.

IT has been long known that an elliptic cylinder is a possible form for the surface of separation between parts of a liquid mass moving rotationally and parts moving irrotationally; and it was, in fact, observed by Kirchhoff that, if the spin is uniform, the stream-function can be proportional to the potential of the cylinder, and then all the conditions are satisfied, provided the cylinder rotates with a definite angular velocity depending on the eccentricity and on the spin. In subsequent investigations of this problem it has been usual to assume the kinematical conditions of continuity of tangential and normal velocities at the surface of separation, and then to show that the pressure equation is identically satisfied; and it can in this way be proved that if ever there exists an elliptic vortex with uniform spin and with no slip at its surface, and if this vortex moves so as to remain elliptic, then it must rotate steadily with the angular velocity given by Kirchhoff's condition (see my paper in *Proc. Lond. Math. Soc.*, Nov., 1893). But it is interesting to enquire whether there can be any steady motion of an elliptic vortex other than that determined by Kirchhoff's equation for the angular velocity, when the condition of continuity of tangential velocity at the surface of the vortex is not satisfied. For the sake of greater generality we may suppose the density of the vortex to be different from that of the surrounding fluid, but, as in Kirchhoff's problem, we shall suppose the spin to be uniform.

Let then the axes of the normal section of the cylinder be a, b, and let the cylinder rotate with angular velocity ω, while the spin or "molecular rotation" of the contained liquid is ζ, the sense of ζ positive being the same as that of ω positive. Also let ρ' be the density of the liquid in rotational motion, and ρ that of the surrounding liquid. Under the

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conditions of no slip at the interface and equal densities, we have Kirchhoff's case expressed by the relations

$$\rho = \rho', \ \omega = 2\zeta ab/(a+b)^2.$$

In addition to this we now determine states of steady motion of the cylinder for which there may be slip at the interface and the densities may be unequal. We find that with a, b, ρ, ρ', ζ given there are two possible values of ω, such that

$$\omega = \frac{2\zeta ab}{a^2 - b^2} \frac{a\gamma - b}{b\gamma + a},$$

where γ is a root of the quadratic

$$\gamma^2[(a^2+b^2)\rho' - 2a^2\rho] - 4\gamma ab(\rho' - \rho) + [(a^2+b^2)\rho' - 2b^2\rho] = 0.$$

We obtain Kirchhoff's case by taking $\rho = \rho'$, $\gamma = 1$. But with equal densities there is another possible steady motion given by $\gamma = -1$, viz. we have in this case

$$\rho = \rho', \ \omega = -2\zeta ab/(a-b)^2,$$

so that the angular velocity is opposite in sense to the spin. This motion is probably unstable. In the general case it is noteworthy that there is no possible steady motion of the kind considered unless $2\rho > \rho'$. In the limiting case when $\rho' = 2\rho$ the quadratic for γ has equal roots, and we have $\gamma = a/b$, $\omega = 2\zeta ab/(a^2+b^2)$, the two values of ω coalescing.

The analysis by which these statements are proved is of a simple character, and only an outline need here be given. The cylinder is referred to its principal axes as axes of x, y; these rotate with angular velocity ω. We denote by ψ', p', ρ', the stream-function, pressure, and density inside the cylinder, by ψ, p, ρ the corresponding quantities outside, and by u, v the velocity components parallel to the axis. We take

$$\psi' = \tfrac{1}{2}(\alpha x^2 + \beta y^2),$$

where $$\nabla^2\psi' = -2\zeta = \alpha + \beta,$$

and since the surface $x^2/a^2 + y^2/b^2 = 1$ always contains the same particles

$$\omega(a^2 - b^2) = -(a^2\alpha - b^2\beta).$$

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Also we take conjugate functions ξ, η, so that

$$x+\iota y = c\cosh(\xi+\iota\eta), \quad c^2 = a^2-b^2.$$

Then $\psi' = \frac{1}{4}c^2[(\alpha\cosh^2\xi+\beta\sinh^2\xi)+\cos 2\eta(\alpha\cosh^2\xi-\beta\sinh^2\xi)]$,

and $\quad \psi = A\xi + Be^{-2\xi}\cos 2\eta$,

where A and B are constants to be determined. The condition of continuity of normal velocity at the interface gives

$$B(a-b)/(a+b) = a^2\alpha - b^2\beta = -\omega(a^2-b^2).$$

The pressure equations inside and outside are easily calculated from the equations of motion. Inside we find

$$-\frac{p'}{\rho'} = \text{const.} + \tfrac{1}{2}x^2\{\alpha\omega-(\alpha+\omega)\beta\} + \tfrac{1}{2}y^2\{\beta\omega-(\beta+\omega)\alpha\}$$

$$= \text{const.} + \tfrac{1}{4}[\omega(\alpha-\beta)(a^2+b^2) - \alpha\beta c^2]\cos 2\eta$$

at the interface.

Outside we find

$$-\frac{p}{\rho} = \text{const.} + u\omega y - v\omega x + \tfrac{1}{2}(u^2+v^2),$$

and, after some reductions, we deduce

$$-\frac{p}{\rho} = \text{const.} + \tfrac{1}{2}h^2\left[\left\{A^2 - \tfrac{1}{4}B^2\left(\frac{a-b}{a+b}\right)^2 - AB\frac{2ab}{(a+b)^2}\right\}\right.$$

$$\left. - \left\{AB - B^2\frac{ab}{(a+b)^2}\right\}\frac{a-b}{a+b}\cos 2\eta + \tfrac{1}{2}B^2\left(\frac{a-b}{a+b}\right)^2\cos^2 2\eta\right]$$

at the interface, where $h = |\,\partial(\xi+\iota\eta)/\partial(x+\iota y)\,|$, and the surface-value of h^2 is given by

$$h^{-2} = \tfrac{1}{2}[a^2+b^2-c^2\cos 2\eta].$$

The term $u\omega y - v\omega x$ is calculated by noting that

$$u = \frac{\omega x}{a^2}h\frac{\partial\phi}{\partial\xi} - \frac{\omega y}{b^2}h\frac{\partial\phi}{\partial\eta}, \quad v = \frac{\omega y}{b^2}h\frac{\partial\phi}{\partial\xi} + \frac{\omega x}{a^2}h\frac{\partial\phi}{\partial\eta}, \quad \omega = -B/(a+b)^2,$$

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ON THE SOLUTION OF LAMÉ'S EQUATION $\frac{d^2U}{du^2} = U\{n(n+1)pu + B\}$ IN FINITE TERMS WHEN $2n$ IS AN ODD NUMBER.

By LAWRENCE CRAWFORD.

TAKE the transformation given by Halphen (*Fonctions Elliptiques*, Vol. II., p. 471),

$$u = 2\theta; \quad z = U(p'\theta)^n,$$

then the equation is

$$\frac{d^2z}{d\theta^2} - 2n\frac{p''\theta}{p'\theta}\frac{dz}{d\theta} + 4z\{n(2n-1)p\theta - B\} = 0.$$

Solve this by a series of descending powers of $p\theta - e_2$, where e_2 is the middle root of the three e_1, e_2, e_3, all three being real in connection with Lamé's equation.

Assume $z = a_t(p\theta - e_2)^t + a_{t-1}(p\theta - e_2)^{t-1} + \ldots + a_1(p\theta - e_2) + a_0$,

then $z = \Sigma a_r(p_0 - e_2)^r$,

therefore we have

$$z' = \Sigma r a_r(p\theta - e_2)^{r-1}p'\theta,$$

and $z'' = \Sigma\{2r(2r+1)a_r(p\theta - e_2)^{r+1} + 12r^2e_2a_r(p\theta - e_2)^r + 2r(2r-1)a_r(e_2 - e_1)(e_2 - e_3)(p\theta - e_2)^{r-1}\}$,

and $p''\theta = 6(p\theta - e_2)^2 + 12e_2(p\theta - e_2) + 6e_2^2 - \frac{1}{2}g_2$,

therefore, substituting in equation, we have

$$\Sigma[\{2r(2r+1)a_r(p\theta - e_2)^{r+1} + 12r^2e_2a_r(p\theta - e_2)^r + r(2r-1)a_r(6e_2^2 - \tfrac{1}{2}g_2)(p\theta - e_2)^{r-1}\} - 2n\{6(p\theta - e_2)^2 + 12e_2(p\theta - e_2) + 6e_2^2 - \tfrac{1}{2}g_2\} ra_r(p\theta - e_2)^{r-1} + 4\{n(2n-1)(p\theta - e_2) - B'\}a_r(p\theta - e_2)^r] = 0,$$

where $B' = B - n(2n-1)e_2$,

i.e. $\Sigma[a_r\{2r(2r+1) - 12nr + 4n(2n-1)\}(p\theta - e_2)^{r+2} + a_r\{12r^2e_2 - 24nre_2 - 4B'\}(p\theta - e_2)^r + a_r(6e_2^2 - \tfrac{1}{2}g_2)\{r(2r-1) - 2nr\}(p\theta - e_2)^{r-1}] = 0$,

therefore equation connecting consecutive a's is

$$a_r(2r-4n)(2r-2n+1) + a_{r+1}\{12(r+1)(r-2n+1)e_2 - 4B'\} + a_{r+2}2(e_2 - e_1)(e_2 - e_3)(r+2)(2r-2n+3) = 0.$$

Equation for t is

$$(2t-4n)(2t-2n+1) = 0,$$

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therefore $t=2n$, or $n-\frac{1}{2}$, both here being possible values; at present we take the latter $t=n-\frac{1}{2}$.

Then highest value of r in equation connecting consecutive a's is $n-\frac{1}{2}$.

Therefore $2r-2n+1$ is negative and so is $2r-4n$; $r-2n+1$ is always negative and $2r-2n+3$ is negative except for $r=n-\frac{1}{2}$ and $r=n-\frac{3}{2}$.

So write equation

$$a_r(2n-r)(2n-1-2r)=a_{r+1}\{6(r+1)(2n-r-1)e_3+2B'\}$$
$$-a_{r+2}(e_1-e_3)(e_2-e_3)(r+2)(2n-2r-3),$$

and as this equation only holds, all terms appearing, if $r<n-\frac{3}{2}$, we see that when a_{r+1} vanishes, a_r and a_{r+2} have opposite signs.

When $r=0$, we get

$$a_0.2n(2n-1)=a_1\{6(2n-1)+2B'\}-a_2(e_1-e_3)(e_2-e_3)2(2n-3),$$

and for $r=-1$, since a_{-1} is to vanish,

$$a_{-1}(2n+1)(2n+1)=0=a_0.2B'-a_1(e_1-e_3)(e_2-e_3)(2n-1),$$

which is the equation for B'.

Take then $a_{n-\frac{1}{2}}$ as a positive constant, then we have a series $a_{n-\frac{1}{2}}$, $a_{n-\frac{3}{2}}$, $a_{n-\frac{5}{2}}$, &c., ..., a_1, a_0, a_{-1} of functions of B' such that for $B'=+\infty$ they have signs $+++\,+\ldots$ and for $B'=-\infty$, signs $+-+-+-\ldots$, thus between $-\infty$ and $+\infty$ we have $n+\frac{1}{2}$ changes of sign lost.

Now for no value of B' can two consecutive a's vanish, for otherwise all would vanish; also for a value of B' which makes a_{r+1} vanish, a_r and a_{r+2} have opposite signs, hence for such a value of B' no change of sign is lost; also $a_{n-\frac{1}{2}}$ is positive and cannot vanish, so we lose a change of sign only when a_{-1} becomes zero, but in all we lose $(n+\frac{1}{2})$ changes, therefore a_{-1} must become zero $(n+\frac{1}{2})$ times as B' goes from $-\infty$ to ∞, *i.e.* $a_{-1}=0$ is an equation with $(n+\frac{1}{2})$ real and different roots, *i.e.* the equation for B' has $(n+\frac{1}{2})$ real and different roots.

Therefore B' has $(n+\frac{1}{2})$ real and different values for our solutions, so we get $(n+\frac{1}{2})$ solutions for U of the type

$$\frac{1}{(p'\theta)^n}[a_{n-\frac{1}{2}}p^{n-\frac{1}{2}}\theta+a'_{n-\frac{3}{2}}p^{n-\frac{3}{2}}\theta+\ldots+a_1'p\theta+a_0'].$$

On expanding powers of $p\theta-e_3$, each solution corresponds to one value of B'.

What now is the second solution corresponding to any one of these? We will find that $t=2n$ will give it. Write

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down equations connecting consecutive α's, taking as solution for z,

$$\alpha_{2n}(p\theta - e_3)^{2n} + \alpha_{2n-1}(p\theta - e_3)^{2n-1} + \ldots + \alpha_1(p\theta - e_3) + \alpha_0.$$

The general equation being

$$\alpha_r(2n-r)(2n-1-2r) - \alpha_{r+1}\{6(r+1)(2n-r-1)e_3 + 2B'\} + \alpha_{r+2}(e_1-e_3)(e_2-e_3)(r+2)(2n-2r-3) = 0,$$

they are

$$\alpha_{2n-1}1.(2n-1) + \alpha_{2n}.2B' = 0,$$

$$\alpha_{2n-2}.2(2n-3) + \alpha_{2n-1}(6.2n - 1.e_3 + 2B') + \alpha_{2n}(e_1-e_3)(e_2-e_3)2n(2n-1) = 0,$$

$$\alpha_{2n-3}.3(2n-5) + \alpha_{2n-2}(6.2.2n - 2.e_3 + 2B') \alpha_{2n-1}(e_1-e_3)(e_2-e_3)(2n-1)(2n-3) = 0,$$

$$\ldots\ldots\ldots

In the first case the equation for B' is got by eliminating α_{2n}, α_{2n-1}, ..., $\alpha_{n+\frac{1}{2}}$ from the $(n+\frac{1}{2})$ equations, so we can write one equation for B' as a determinant

$$0 = \begin{vmatrix} 2B' & , & 2n-1 & , & 0 & , & 0 & , \ldots\ldots, & \vdots & , & \vdots \\ (e_1-e_3)(e_2-e_3)2n(2n-1), & & 2B'+6(2n-1)e_2 & , & 2(2n-3) & , & 0 & , \ldots\ldots, & \vdots & , & \vdots \\ 0 & , & (e_1-e_3)(e_2-e_3)(2n-1)(2n-3), & & 2B'+12(2n-2)e_2, & & 3(2n-5), & \ldots\ldots, & \vdots & , & \vdots \\ \vdots & , & \vdots & , & \vdots & , & \vdots & , \ldots\ldots, & \vdots & , & \vdots \\ \vdots & , & \vdots & , & \vdots & , & \vdots & , \ldots\ldots, & (e_1-e_3)(e_2-e_3)(2n+3), & & 2B'+6(n+\tfrac{1}{2})(n-\tfrac{1}{2})e_2 \end{vmatrix}.$$

Multiply now first column by $(e_1-e_3)(e_2-e_3)$, second column by it squared, third by it cubed, and so on, divide first row by it, second row by it squared, and so on, then our equation is

$$\begin{vmatrix} 2B' & , (2n-1)(e_1-e_3)(e_2-e_3), & 0 & , & 0 & , \ldots\ldots, & \vdots & , & \vdots \\ 2n(2n-1), & 2B'+6(2n-1)e_2 & , 2(2n-3)(e_1-e_3)(e_2-e_3), & & 0 & , \ldots\ldots, & \vdots & , & \vdots \\ 0 & , (2n-1)(2n-3) & , 2B'+12(2n-2)e_2 & , & 3(2n-5)(e_1-e_3)(e_2-e_3), & \ldots\ldots, & \vdots & , & \vdots \\ \vdots & , \vdots & , \vdots & , & \vdots & , \ldots\ldots, & \vdots & , & \vdots \\ \vdots & , \vdots & , \vdots & , & \vdots & , \ldots\ldots, & 2n+3, & & 2B'+6(n+\tfrac{1}{2})(n-\tfrac{1}{2})e_2 \end{vmatrix},$$

but our equation for B' in other case is got by taking the latter $(n+\frac{1}{2})$ equations on last page, writing a for α and then eliminating a's, *i.e.* same as eliminating α's from latter $(n+\frac{1}{2})$ equations, but if we consider quantities to be eliminated as $\alpha_0, -\alpha_1, \alpha_2, -\alpha_3$, &c., and eliminate by a determinant, we find we get exactly same determinantal equation as the latter of the two above.

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So our equations for B' are identical, and thus for each value of B' we have two solutions for z, and therefore the complete solution of the equation.

The complete solution of $\frac{d^2U}{du^2} = U\{n(n+1)pu + B\}$ then for $2n$ an odd integer is, where $n = 2\theta$,

$$U = \frac{1}{(p'\theta)^n}[A\{(p\theta - e_2)^{2n} + a_{2n-1}(p\theta - e_2)^{2n-1} + \ldots + a_{n+\frac{1}{2}}(p\theta - e_2)^{n+\frac{1}{2}}\}$$

$$+ B\{(p\theta - e_2)^{n-\frac{1}{2}} + a_{n-\frac{3}{2}}(p\theta - e_2)^{n-\frac{3}{2}} + \ldots + a_1(p\theta - e_2) + a_0\}].$$

Take the latter of the two solutions

$$U = \frac{1}{(p'\theta)^n}\{(p\theta - e_2)^{n-\frac{1}{2}} + a_{n-\frac{3}{2}}(p\theta - e_2)^{n-\frac{3}{2}} + \ldots\} :$$

call this $f(\theta)$.

Now equation for U is unaltered by writing $u + 2\omega$ for u, *i.e.* $\theta + \omega$ for θ, therefore if $U = f(\theta)$ be a solution, $U = f(\theta + \omega)$ must also be one, 2ω being any period of u.

Take here ω'', where $p\omega'' = e_2$.

We have

$$\{p(\theta + \omega'') - e_2\}\{p\theta - e_2\} = (e_2 - e_1)(e_2 - e_3),$$

and
$$p'(\theta + \omega'') = -\frac{(e_2 - e_1)(e_2 - e_3)p'\theta}{(p\theta - e_2)^2},$$

therefore $f(\theta + \omega'') = \frac{(p\theta - e_2)^{2n}}{(e_1 - e_2)^n (e_2 - e_3)^n (p'\theta)^n}$

$$\times \left\{\frac{(e_2 - e_1)^{n-\frac{1}{2}}(e_2 - e_3)^{n-\frac{1}{2}}}{(p\theta - e_2)^{n-\frac{1}{2}}} + a_{n-\frac{3}{2}}\frac{(e_2 - e_1)^{n-\frac{3}{2}}(e_2 - e_3)^{n-\frac{3}{2}}}{(p\theta - e_2)^{n-\frac{3}{2}}} + \ldots + a_0\right\}$$

$$= \frac{1}{(e_1 - e_2)^n (e_2 - e_3)^n (p'\theta)^n}\{a_0(p\theta - e_2)^{2n} - a_1(e_1 - e_2)(e_2 - e_3)(p\theta - e_2)^{2n-1}$$

$$+ a_2(e_1 - e_2)^2(e_2 - e_3)^2(p\theta - e_2)^{2n-2} + \ldots$$

$$+ (-1)^{n-\frac{3}{2}}(e_1 - e_2)^{n-\frac{1}{2}}(e_2 - e_3)^{n-\frac{3}{2}} a_{n-\frac{3}{2}}(p\theta - e_2)^{n+\frac{1}{2}}$$

$$+ (-1)^{n-\frac{1}{2}}(e_1 - e_2)^{n-\frac{1}{2}}(e_2 - e_3)^{n-\frac{1}{2}} a_{n-\frac{1}{2}}(p\theta - e_2)^{n+\frac{1}{2}}\},$$

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but if we compare $a_{2n}, a_{2n-1}, \ldots, a_{n+\frac{1}{2}}$ and $a_0, a_1, \ldots, a_{n-\frac{1}{2}}$ from our equation for α's, replacing α's by a's, we see the expression inside brackets is proportional to

$$(p\theta - e_3)^{2n} + a_{2n-1}(p\theta - e_3)^{2n-1} + \ldots + a_{n+\frac{1}{2}}(p\theta - e_3)^{n+\frac{1}{2}},$$

therefore $f(\theta + \omega'')$ only gives us again the other solution, formerly derived in a different way. Thus, we have two and only two independent solutions, for on increasing θ in $f(\theta + \omega'')$ by ω'' again, we get $f(\theta + 2\omega'')$, *i. e.* $f(\theta)$ again, because $f(\theta)$ is periodic in $2\omega''$.

No fresh solution is obtained by getting one solution and replacing in it θ by $\theta + \omega$ or $\theta + \omega'$, for if we solve in powers of $p\theta - e_1$ or $p\theta - e_2$ respectively (and we know already that the resulting equation for B must have $(n + \frac{1}{2})$ real roots), we find as above we have two and only two independent solutions.

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INTRANSITIVE SUBSTITUTION GROUPS OF TEN LETTERS.

By Geo. A. Miller, Ph.D.

IN Vol. xxv. of this *Journal* Professor Cayley gave, in a condensed form and with some additions, the lists of the substitution groups through eight letters, which had been obtained by Serret and Mr. Askwith. In Vol. xxvi. Professor Cole gave a supplementary list, together with the groups of nine letters, and in the first number of this volume a list of the transitive groups of ten and eleven letters.

In the following list of the intransitive groups of ten letters I have followed, in the main, the notation explained by Professor Cayley, pp. 71–77, Vol. xxv., of this *Journal*. On account of the very large number of groups and the varied correspondences, it became necessary to deviate from it in a few particulars which I proceed to explain.

Whenever a correspondence is indicated, all the groups which can be found in this way are included, *e.g.*

$$\{(abcdef)_{12}(ghij)_{a}\} \text{ dim.}$$

gives rise to nine distinct groups, since each of the given groups has three subgroups of half its order. These nine groups have therefore been written thus:

$$1, 9\ \{(abcdef)_{12}(ghij)_{a}\} \text{ dim.}$$

In two instances, under the system of intransitivity 2, 2, 2, 2, 2, some of the groups obtained by dimidiating are already in the list, and in these cases the special subgroups used to obtain the groups that are meant are separately given after the respective groups. By this means it became possible to give to the term dimidiate and to the various correspondences perfectly definite meanings. All correspondences, except those expressed by the term dimidiate and the notation explained in the following paragraph, are indicated by the subscript a, b, where a and b represent the number of substitutions in the first and second groups respectively, which are placed in correspondence. Correspondences are always to be taken with respect to the last group; *i.e.* all the part up to the last group is to be placed in correspondence with the last group.

As in the former lists a 1, 1 correspondence between the

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substitutions of the same group in different letters is represented by placing a period between the two systems of letters. For instance, $(abcde.fghij)$ cyc. means the group

$$1,$$
$$abcde.fghij,$$
$$acebd.fhjgi,$$
$$adbec.figjh,$$
$$aedcb.fjihg.$$

When a group can have more than one such 1, 1 correspondence, this notation does not distinguish between the groups thus obtained; $(abcd.efgh)_8$ may, for instance, mean either of the following groups:

$1,$	$1,$
$ac.eg,$	$ac.ef.gh,$
$bd.fh,$	$bd.eh.fg,$
$ac.bd.eg.fh,$	$ac.bd.eg.fh,$
$ab.cd.ef.gh,$	$ab.cd.eg,$
$ad.bc.eh.fg,$	$ad.bc.fh,$
$abcd.efgh,$	$abcd.ehgf,$
$adcb.ehgf,$	$adcb.efgh.$

Where it seemed necessary to distinguish between these two groups for the sake of clearness $(abcd.efgh)_{8_1}$ has been used to represent the former, and $(abcd.efgh)_{8_2}$ to represent the latter. This is the only case in which a special notation of this kind seemed necessary.

In examining into all the possible correspondences between two groups, which are to be used to form new groups, the well-known principle that similar subgroups may be placed in correspondence is of constant use; but it is necessary to observe that this applies to the corresponding divisions of the groups and not to the groups as a whole, *e.g.* in the following 2, 1 correspondence between $(abcdef)_{24_5}$ and $(ghij)$ pos.,

$1,$	$ac.gh.ij,$	$abe.cdf.ghi,$	$aeb.cfd.gih,$
$ac.bd.ef,$	$bd.ef.gh.ij,$	$adecbf.ghi,$	$afbced.gih,$

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bd.gj.hi	*abf.cde.gij,*	*afb.ced.gji,*
ac.ef.gj.hi,	*adfcbe.gij,*	*aebcfd.gji,*
ef.gi.hj,	*ade.bfc.hji,*	*aed.bcf.hij,*
ac.bd.gi.hj,	*abecdf.hji,*	*afdceb.hij,*
	adf.bec.gjh,	*afd.bce.ghj,*
	abfcde.gjh,	*aedcfb.ghj,*

the self-conjugate subgroups of orders two and eight in the former correspond to those of orders one and four respectively in the latter group. But the former group has two even self-conjugate subgroups, viz. those of orders four and twelve, while the latter does not have any self-conjugate subgroups of half these orders.

In finding the self-conjugate subgroups of a given group, with a view to determining a correspondence, it is not sufficient to find all the groups of the required orders and of degrees not higher than the degree of the group under consideration and to determine which of these are self-conjugate subgroups. For the division of a group according to similar self-conjugate subgroups may, by an *a*, *b* correspondence, give rise to groups which are not similar. For example, the group $(abcdef)_{18}$ will be divided in the following ways by using the two similar groups of (*abc.def*) cyc. and (*abc.dfe*) cyc. as bases of division

1,	1,
abc.def,	*abc.dfe,*
acb.dfe,	*acb.def,*
abc,	*abc,*
dfe,	*def,*
acb.def,	*acb.dfe,*
acb,	*acb,*
def,	*dfe,*
abc.dfe,	*abc.def,*
ad.be.cf,	*ad.be.cf,*
aecdbf,	*af.bd.ce,*
afbdce,	*ae.bf.cd,*

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$ae.bf.cd,$	$aecdbf,$
$afcebd,$	$adcfbe,$
$adbecf,$	$afcebd,$
$af.bd.ce,$	$afbdce,$
$adcfbe,$	$aebfcd,$
$aebfcd,$	$adbecf.$

These two modes of division taken separately show that $(abcdef)_{18}$ is isomorphic with each of the types of groups of order 6; viz. (abc) all and $(abcdef)$ cyc.

The division of the list, according to its systems of intransitivity, is believed to add much to its usefulness. This arrangement is due to a suggestion by Professor Cole, to whom I am under great obligations for other helpful suggestions while preparing this list, as well as for the verification of some of my work.

I take this opportunity to give a non-primitive group* of degree eight and order twenty-four, which does not appear in the lists which I consulted. It may be written thus

$$(abcd.efgh) \text{ pos. } (ae.bf.ch.dg).$$

The substitutions are

1,	$ab.cd.ef.gh,$	$abc.efg,$	$afch.bgde,$
	$ac.bd.eg.fh,$	$abd.efh,$	$ahcf.bedg,$
	$ad.bc.eh.fg,$	$acd.egh,$	$afdg.bhce,$
	$af.be.cg.dh,$	$bcd.fgh,$	$agdf.bech,$
	$ag.bf.ce.dh,$	$bdc.fhg,$	$agbh.cfde,$
	$ah.bf.cg.de,$	$acb.egf,$	$ahbg.cedf,$
	$ae.bg.cf.dh,$	$adb.ehf,$	
	$ae.bh.cg.df,$	$adc.ehg,$	
	$ae.bf.ch.dg.$		

* The group of order 1344 and degree 8 is not found in either of the lists referred to in the first paragraph of this article. It contains all the substitutions which transform A $(abcdefgh)_8$ into itself and may be written thus

$$+[\{(ab)(cd)(eh)(fg)\} \text{ pos. } (ABCD) \text{ all}] \ (abcefdg) \text{ cyc.}$$

This group is found in Kirkman's list of the transitive groups through degree 10 in *Proceedings of the Literary and Philosophical Society of Manchester*, Vol. III. It is also given by Jordan in *Comptes Rendus*, Vol. LXXIII. The two intransitive groups of order six, $(ab.cd.ef)$ (ghi) cyc. and $\{(abcdef) \text{ cyc. } (ghi) \text{ cyc.}\}_{2,1}$ are not found in the list of groups of nine letters to which I referred in the first paragraph. I worked through this list and found no other errors.

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It need scarcely be added that the groups which I have employed in the formation of new groups are written in the form in which they occur in the lists referred to, and that the groups of the same order are numbered from the first to the last, regardless of the division according to systems of intransitivity.

Systems of intransitivity 5, 5.

GROUPS.

Order.	No.	
5.	1	$(abcde.fghij)$ cyc.,
10.	1	$(abcde.fghij)_{10}$,
20.	1	$(abcde.fghij)_{20}$,
25.	1	$(abcde)$ cyc. $(fghij)$ cyc.,
50.	1, 2	$(abcde)$ cyc. $(fghij)_{10}$, $\{(abcde)_{10}(fghij)_{10}\}$ dim.
60.	1	$(abcde.fghij)$ pos.,
100.	1, 2	$(abcde)_{20}(fghij)$ cyc., $(abcde)_{10}(fghij)_{10}$,
	3	$\{(abcde)_{20}(fghij)_{10}\}$ dim.
	4, 5	$\{(abcde)_{20}(fghij)_{20}\}_{5,5}$,
120.	1	$(abcde.fghij)$ all,
200.	1, 2	$(abcde)_{20}(fghij)_{10}$, $\{(abcde)_{20}(fghij)_{20}\}$ dim.,
300.	1	$(abcde)$ pos. $(fghij)$ cyc.,
400.	1	$(abcde)_{20}(fghij)_{20}$,
600.	1, 2	$(abcde)$ all $(fghij)$ cyc., $(abcde)$ pos. $(fghij)_{10}$,
	3	$\{(abcde)$ all $(fghij)_{10}\}$ dim.,
1200.	1, 2	$(abcede)$ all $(fghij)_{10}$, $(abcde)$ pos. $(fghij)_{20}$,
	3	$\{(abcde)$ all $(fghij)_{20}\}$ dim.
2400.	1	$(abcde)$ all $(fghij)_{20}$,
3600.	1	$(abcde)$ pos. $(fghij)$ pos.,
7200.	1	$(abcde)$ pos. $(fghij)$ all,
	2	$\{(abcde)$ all $(fghij)$ all$\}$ dim.,
14400.	1	$(abcde)$ all $(fghij)$ all.
	28.	

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	10	$\{(abcdef)_{18}\,(ghij)\ \text{pos}\}_{6,4}$,
	11, 16	$[\{(abcdef)_{36},\ (abcdef)_{36_2}\}\ \{(ghij)_4,\ (ghij)\ \text{cyc.}\}]\ \text{dim.}$
	17, 19	$\{(abcdef)_{36_3}\,(ghij)_8\}_{9,2}$,
	20	$\{(abcdef)_{72}\,(ghij)_4\}_{18,1}$,
	21, 22	$\{(abcdef)_{72}\,(ghij)_8\}_{9,1}$,
	23	$\{(abcdef)_{18}\,(ghij)\ \text{all}\}_{3,4}$,
96.	1, 2	$\{(abcdef)_{12},\ (abcdef)_{12_2}\}\ (ghij)_8$,
	3, 3	$\{(\pm abcdef)_{24},\ (+abcdef)_{24},\ (abcdef)_{24_{45}}\}\ \{(ghij)_4,\ (ghij)\ \text{cyc.}\}$,
	9, 17	$[\{(\pm abcdef)_{24},\ (+abcdef)_{24},\ (abcdef)_{24_5}\}\ (ghij)_8]\ \text{dim.}$
	18	$\{(abcdef)_{24_5}\,(ghij)\ \text{pos.}\}_{3,4}$,
	19, 20	$[\{(\pm abcdef)_{24}\ (+abcdef)_{24}\}\ (ghij)\ \text{all}]_{4,3}$,
	21, 26	$[(abcdef)_{48}\ \{(ghij)_4,\ (ghij)\ \text{cyc.}\}]\ \text{dim.}$
	27, 32	$\{(abcdef)_{48}\,(ghij)_8\}_{12,2}$,
144.	1, 2	$\{(abcdef)_6,\ (abcdef)\ \text{cyc.}\}\ (ghij)\ \text{all}$,
	3, 5	$\{(abcdef)_{12}\,(ghij)\ \text{all}\}\ \text{dim.}$
	6, 7	$\{(abcdef)_{12},\ (abcdef)_{12_2}\}\ (ghij)\ \text{pos.}$
	8	$(abcdef)_{18}\,(ghij)_8$,
	9, 12	$\{(abcdef)_{36},\ (abcdef)_{36_2}\}\ \{(ghij)_4,\ (ghij)\ \text{cyc.}\}$,
	13, 21	$[\{(abcdef)_{36},\ (abcdef)_{36_3}\}\ (ghij)_8]\ \text{dim.}$
	22	$\{(abcdef)_{36_2}\,(ghij)\ \text{all}\}_{6,4}$,
	23, 28	$[(abcdef)_{72}\ \{(ghij_4,\ (ghij)\ \text{cyc.}\}]\ \text{dim.}$
	29, 34	$\{(abcdef)_{72}\,(ghij)_8\}_{18,2}$,
192.	1, 3	$\{(\pm abcdef)_{24},\ (+abcdef)_{24},\ (abcdef)_{24}\}\ (ghij)_8$,
	4, 5	$(abcdef)_{48}\ \{(ghij)_4,\ (ghij)\ \text{cyc.}\}$,
	6, 14	$\{(abcdef)_{48}\,(ghij)_8\}\ \text{dim.}$
	15	$\{(abcdef)_{48}\,(ghij)\ \text{all}\}_{8,4}$,

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216. 1, 2 $(abcdef)_{18}(ghij)$ pos., $\{(abcdef)_{18}(ghij)$ all$\}$ dim.

240. 1, 2 $[(abcdef)_{120}\{(ghij)_4, (ghij)$ cyc.$\}]$ dim.

3, 4 $(abcdef)_{60}\{(ghij)_4, (ghij)$ cyc.$\}$,

288. 1, 2 $\{(abcdef)_{12}, (abcdef)_{12_2}\}(ghij)$ all,

3, 5 $\{(\pm abcdef)_{24}, (+abcdef)_{24}, (abcdef)_{24_2}\}(ghij)$ pos.

6, 8 $[\{(\pm abcdef)_{24}, (+abcdef)_{24}, (abcdef)_{24_2}\}(ghij)$ all$]$ dim.

9, 10 $\{(abcdef)_{36}, (abcdef)_{36_2}\}(ghij)_3$,

11, 12 $(abcdef)_{72}\{(ghij)_4, (ghij)$ cyc.$\}$,

13, 21 $\{(abcdef)_{72}(ghij)_8\}$ dim.

384. 1 $(abcdef)_{48}, (ghij)_8$,

432. 1 $(abcdef)_{18}, (ghij)$ all,

2, 3 $\{(abcdef)_{36}, (abcdef)_{36_2}\}(ghij)$ pos.

4, 6 $[\{(abcdef)_{36}, (abcdef)_{36_2}\}(ghij))$ all$]$ dim.

480. 1, 2 $(abcdef)_{120}\{(ghij)_4, (ghij)$ cyc.$\}$,

3, 6 $\{(abcdef)_{120}(ghij)_6\}$ dim., $(abcdef)_{60}(ghij)_6$,

576. 1, 3 $\{(\pm abcdef)_{24}, (+abcdef)_{24}, (abcdef)_{24}\}(ghij)$ all,

4, 6 $\{(abcdef)_{48}(ghij)$ all$\}$ dim.

7, 8 $(abcdef)_{48}(ghij)$ pos., $(abcdef)_{72}(ghij)_8$,

720. 1 $(abcdef)_{60}(ghij)$ pos.

864. 1, 2 $\{(abcdef)_{36}, (abcdef)_{36_2}\}(ghij)$ all,

3, 6 $(abcdef)_{72}(ghij)$ pos., $\{(abcdef)_{72}(ghij)$ all$\}$ dim.

960. 1 $(abcdef)_{120}(ghij)_8$,

1152. 1 $(abcdef)_{48}(ghij)$ all,

1440. 1, 2 $(abcdef)_{120}(ghij)$ pos., $\{(abcdef)_{120}(ghij)$ all$\}$ pos.

3, 5 $(abcdef)$ pos. $\{(ghij)_4, (ghij)$ cyc.$\}$, $(abcdef)_{60}(ghij)$ all,

6, 7 $[abcdef)$ all $\{(ghij)_4, (ghij)$ cyc.$\}]$ dim.

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1728.	1	$(abcdef)_{72}$ $(ghij)$ all,
2880.	1, 2	$(abcdef)_{120}$ $(ghij)$ all, $(abcdef)$ pos. $(ghij)_8$,
	3, 4	$(abcdef)$ all $\{(ghij)_4$, $(ghij)$ cyc.$\}$,
	5, 7	$\{(abcdef)$ all $(ghij)_8\}$ dim.
4320.	1	$(abcdef)$ pos. $(ghij)$ pos.
5760.	1	$(abcdef)$ all $(ghij)_8$.
8640.	1, 2	$(abcdef)$ all $(ghij)$ pos., $(abcdef)$ pos. $(ghij)$ all,
	3	$\{(abcdef)$ all $(ghij)$ all$\}$ dim.
17280.	1	$(abcdef)$ all $(ghij)$ all.

244.

Systems of intransitivity 7, 3.

21.	1, 2	$(abcdefg)$ cyc. (hij) cyc., $\{(abcdefg)_{21}$ (hij) cyc.$\}_{7,1}$,
42.	1, 2	$(abcdefg)$ cyc. (hij) all, $\{(abcdefg)_{14}$ (hij) all$\}$ dim.
	3, 4	$(abcdefg)_{14}$ (hij) cyc., $\{(abcdefg)_{42}$ (hij) cyc$\}_{14,1}$
63.	1	$(abcdefg)_{21}$ (hij) cyc.
84.	1	$(abcdefg)_{14}$ (hij) all,
126.	1, 2	$(abcdefg)_{21}$ (hij) all, $(abcdefg)_{42}$ (hij) cyc.
	3	$\{(abcdefg)_{42}$ (hij) all$\}$ dim.
252.	1	$(abcdefg)_{42}$ (hij) all,
504.	1	$(abcdefg)_{168}$ (hij) cyc.
1008.	1	$(abcdefg)_{168}$ (hij) all,
7560.	1	$(abcdefg)$ pos. (hij) cyc.
15120.	1, 2	$(abcdefg)$ pos. (hij) all, $\{(abcdefg)$ all (hij) all$\}$ dim.
	3	$(abcdefg)$ all (hij) cyc.
30240.	1	$(abcdefg)$ all (hij) all.

19.

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Systems of intransitivity 8, 2.

8. 1, 2 $[\{A\,(abcdefgh)_8, D\,(abcdefgh)_8\}\,(ij)$ dim.

3, 6 $[\{B\,(abcdefgh)_8, C\,(abcdefgh)_8\}\,(ij)]$ dim.

7 $\{(abcdefgh)$ cyc. $(ij)\}$ dim.

16. 1, 5 $\{A\,(abcdefgh)_8, D\,(abcdefgh)_8, B\,(abcdefgh)_8,$
$C\,(abcdefgh)_8, (abcdefgh)$ cyc.$\}\,(ij)$,

6, 10 $\{ae.bf.cg.dh)\,B'\,(abcd.efgh)_8\,(ij)\}$ dim.

11, 15 $\{(ae.bf.cg.dh)\,C'\,(abcd.efgh)_8\,(ij)\}$ dim.

16, 17 $\{(aebf.cdgh)$ cyc. $(ac.bd)\,(ef.gh)\,(ij)\}$ dim.

18, 19 $\{(aebfcgdh)$ cyc. $(1, ac.bd)\,(ij)\}$ dim.

20, 25 $[(abcd.efgh)_{8_2}\,\{(aebfcgdh), (afbgchde)\}\,(ij)]$ dim.

24. 28, 29 $[(abcd.efgh)$ pos. $\{(ae.bf.cg.dh),$
$(ae.bf.ch.dg)\}\,(ij)]$ dim.

32. 1, 6 $[(ae.bf.cg.dh)\,B'\,(abcd.efgh)_8,$
$(ae.bf.cg.dh)\,C'\,(abcd.efgh)_8,$
$(aebf.cdgh)$ cyc. $(ac.bd)\,(ef.gh),$
$(aebfcgdh)$ cyc. $(1, ac.bd),$
$(abcd.efgh)_{8_2}\,\{(aebfcgdh), (afbgchde)\}]\,(ij)$,

7, 9 $[\{(abcd)_8\,(efgh)_8\}_{2,2}\,(ae.bf.cg.dh)\,(ij)]$ dim.

10, 11 $[\{(abcd)_8\,(efgh)_8\}_{1,2}\,(aebf.cgdh)\,(ij)]$ dim.

12, 15 $[\{(abcd)_8\,(efgh)_8\}_{2,2}\,\{(aecg.bf.dh),$
$(aebfcgdh)\}\,(ij)]$ dim.

16, 20 $[\{(abcd)_8\,(efgh)_8\}_{2,2_2}\,(ae.bf.cg.dh)\,(ij)]$ dim.

21, 23 $[\{(abcd)_8\,(efgh)_8\}_{2,2_3}\,(ae.bf.cg.dh)\,(ij\,]$ dim.

24, 26 $[(abcd)$ cyc. $(efgh)$ cyc. $(ae.bf.cg.dh)\,(ij)$ dim.

27, 29 $\{(abcd)_4\,(efgh)_4\,(ae.bf.cg.dh)\,(ij)\}$ dim.

48. 26, 28 $[(abcd.efgh)$ pos. $\{(ae.bf.cg.dh), (ae.bf.ch.dg)\},$
$(ab.cd.ef.gh)\,(ABCD)$ pos.$]\,(ij)$,

29, 32 $\{(abcdefgh)_{48}\,(ij)\}$ dim.,
$\{(ab.cd.ef.gh)\,(ABCD)$ all $(ij)\}$ dim.

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64. 1, 4 $[\{(abcd)_8\,(efgh)_8\}_{2,2}\,\{(ae.bf.cg.dh), (aebf.cgdh),$
$(aecg.bf.dh), (aebfcgdh)\}]\,(ij),$

5, 6 $[\{(abcd)_8\,(efgh)\}_{2,2_2},$
$\{(abcd)_8\,(efgh)_8\}_{2,2_2}]\,(ae.bf.cg.dh)\,(ij),$

7, 8 $\{(abcd)$ cyc. $(efgh)$ cyc.,
$(abcd)_4\,(efgh)_4\}\,(ae.bf.cg.dh)\,(ij),$

9, 11 $[\{(abcd)_8$ cyc. $(efgh)_8$ cyc.$\}$ dim.
$(aecg.bf.dh)\,(ij)]$ dim.

12, 14 $[\{(abcd)_8$ pos. $(efgh)_8$ pos.$\}$ dim.
$(aebfcgdh)\,(ij)]$ dim

15, 17 $[\{(abcd)_8$ com. $(efgh)_8$ com.$\}$ dim.
$(ae.bf.cg.dh)\,(ij)]$ dim.

18, 20 $[\{(abcd)_8$ com. $(efgh)_8$ com.$\}$ dim.
$(aebfcgdh)\,(ij)]$ dim.

21, 25 $[M\,\{(abcd)_8\,(efgh)_8\}$ dim. $(ae.bf.cg.dh)\,(ij)]$ dim.

26, 30 $[N\{(abcd)_8\,(efgh)_8\}$ dim. $(ae.bf.cg.dh)\,(ij)]$ dim.

96. 33, 34 $\{(abcdefgh)_{48}, (ab.cd.ef.gh)\,(ABCD)$ all$\}\,(ij),$

35 $[(ae.bf.cg.dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris. $(ij)]$ dim.

36 $[(ae.bg.cf.dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris $(ij)]$ dim.

112. 1 $\{A\,(abcdefgh)_8\,(bcedghf)$ cyc. $(ij)\}$ dim.

128. 1 $M\{(abcd)_8\,(efgh)_8\}$ dim. $(ae.bf.cg.dh)\,(ij),$

2 $N\{(abcd)_8\,(efgh)_8\}$ dim. $(ae.bf.cg.dh)\,(ij),$

3 $\{(abcd)_8$ cyc. $(efgh)_8$ cyc.$\}$ dim. $(aecg.bf.dh)\,(ij),$

4 $\{(abcd)_8$ com. $(efgh)_8$ com$\}$ dim. $(ae.bf.cg.dh)\,(ij),$

5, 6 $\{(abcd)_8$ pos. $(efgh)_8$ pos.$\}$ dim.,
$\{(abcd)_8$ com. $(efgh)_8$ com.$\}\,(aebfcgdh)\,(ij)$

7, 13 $[\{(abcd)_8\,(efgh)_8\}\,(ae.bf.cg.dh)\,(ij)]$ dim.

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192. 16 $\{(ab)\,(cd)\,(ef)\,(gh)\}$ pos. $(ABCD)$ pos. (ij),

17, 18 $\{(abcd)_{12}\,(efgh)_{12}\}$ tris. $\{(ae.bf.cg.dh),$
$(ae.bg.cf.dh)\}\,(ij)$,

19, 21 $[\{(abcd)\text{all}\,(efgh)\,\text{all}\}$ sex.$(ae.bf.cg.dh)(ij)]$ dim.

22 $[\{(ab)\,(cd)\,(ef)\,(gh)\}$ all $(ABCD)$ pos. $(ij)]$ dim.

23 $\{+[\{(ab)\,(cd)\,(ef)\,(gh)\}$ pos.
$\times\,(ABCD)$ all$]\,(ij)\}$ dim.

24 $\{\pm[\{(ab)\,(cd)\,(ef)\,(gh)\}$ pos.
$(ABCD)$ all$]\,(ij)\}$ dim.

256. 1 $\{(abcd)_8\,(efgh)_8\}\,(ae.bf.cg.dh)\,(ij)$,

288. 22 $\{(abcd)$ pos. $(efgh)$ pos. $(ae.bf.cg.dh)\,(ij)\}$ dim.

336. 1, 2 $\{(abcdefgh)_{168},\ A\,(abcdefgh)_8\,(bcedghf)_{21}\}\,(ij)$,

3 $\{(abcdefgh)_{336}\,(ij)\}$ dim.

384. 2 $\{(abcd)$ all $(efgh)$ all$\}$ sex. $(ae.bf.cg.dh)\,(ij)$,

3 $\{(ab)\,(cd)\,(ef)\,(gh)\}$ all $(ABCD)$ pos. (ij),

4 $+[\{(ab)\,(cd)\,(ef)\,(gh)\}$ pos. $(ABCD)$ all$]\,(ij)$,

5 $\pm[\{(ab)\,(cd)\,(ef)\,(gh)\}$ pos. $(ABCD)$ all$]\,(ij)$,

6, 8 $[(1,\ ag.bh,\ cf.dh,\ eg.fh)$
$\times\,\{(abcdef)_{48}\,(gh)\}\,(ij)]$ dim.

576. 9 $(abcd)$ pos. $(efgh)$ pos. $(ae.bf.cg.dh)\,(ij)$,

10, 11 $[(ae.bf.cg.dh)\,\{(abcd)$ all
$\times\,(efgh)$ all$\}$ pos. $(ij)]$ dim.

12 $[\{(abcd)$ all $(efgh)$ all$\}$ pos. $(aebf.cg.dh)\,(ij)]$ dim.

672. 1 $(abcdefgh)_{336}\,(ij)$,

768. 1 $(1,\ ag.bh,\ cf.dh,\ eg.fh)\,\{(abcdef)_{48}\,(gh)\}\,(ij)$,

1152. 2, 3 $\{(abcd)$ all $(efgh)$ all$\}$ pos. $\{(ae.bf.cg.dh),$
$(aebf.cg.dh)\}\,(ij)$,

4, 6 $\{(abcd)$ all $(efgh)$ all $(ae.bf.cg.dh)\,(ij)\}$ dim.

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2304. 1 $(ae.bf.cg.dh)\ \{(abcd)\ \text{all}\ (efgh)\ \text{all}\}\ (ij)$,

2688. 1 $+\ [\{(ab)\ (cd)\ (eh)\ (fg)\}\ \text{pos.}$
$\times\ (ABCD)\ \text{all}]\ (abcefdg)\ \text{cyc.}\ (ij)$,

40320. 1, 2 $(abcdefgh)\ \text{pos.}\ (ij)$, $\{(abcdefgh)\ \text{all}\ (ij)\}\ \text{dim.}$

80640. 1 $(abcdefgh)\ \text{all}\ (ij)$.

155.

Systems of intransitivity 4, 4, 2.

4. 1, 2 $\{(abcd.efgh)_4\ (ij)\}\ \text{dim.}$,
$\{(abcd.efgh)\ \text{cyc.}\ (ij)\}\ \text{dim.}$

8. 8, 9 $(abcd.efgh)_4\ (ij)$, $(abcd.efgh)\ \text{cyc.}\ (ij)$,

10, 14 $\{[\{(abcd)\ \text{cyc.}\ (efgh)\ \text{cyc.}\}\ \text{pos.},$
$\{(abcd)_4\}\ (efgh)_4\ \text{dim.}]\ (ij)\}\ \text{dim.}$

15, 16 $[\{(abcd)\ \text{cyc.}\ (efgh)_4\}\ \text{dim.}\ (ij)]\ \text{dim.}$

17, 21 $\{(abcd.efgh)_8\ (ij)\}\ \text{dim.}$

22, 24 $[\{(abcd)_8\ (efgh)_4\}_{2,1}\ (ij)]\ \text{dim.}$

16. 26, 28 $[\{(abcd)_4,\ (abcd)\ \text{cyc.}\}\ (efgh)\ \text{cyc.},$
$(abcd)_4\ (efgh)_4]\ \text{dim.}\ (ij)$,

29, 31 $[(abcd.efgh)_8,\ \{(abcd)_8\ (efgh)_4\}_{2,1}]\ (ij)$,

32, 35 $[\{(abcd)\ \text{cyc.}\ (efgh)\ \text{cyc.}, (abcd)_4\ (efgh)_4\}\ (ij)]\ \text{dim.}$

36, 38 $\{(abcd)\ \text{cyc.}\ (efgh)_4\ (ij)\}\ \text{dim.}$

39, 53 $[\{(abcd)_8\ (efgh)_4\}\ \text{dim.}\ (ij)]\ \text{dim.}$

54, 62 $[\{(abcd)_8\ (efgh)\ \text{cyc.}\}\ \text{dim.}\ (ij)]\ \text{dim.}$

63, 76 $[\{(abcd)_8\ (efgh)_8\}_{2,2}\ (ij)]\ \text{dim.}$

24. 30, 31 $(abcd.efgh\}\ \text{pos.}\ (ij)$, $\{(abcd.efgh)\ \text{all}\ (ij)\}\ \text{dim.}$

32. 30, 32 $[\{(abcd)_4,\ (abcd)\ \text{cyc.}\}\ (efgh)\ \text{cyc.},$
$(abcd)_4\ (efgh)_4]\ (ij)$,

33, 38 $[(abcd)_8\ \{(efgh)_4,\ (efgh)\ \text{cyc.}\}\ \text{dim.}\ (ij)$,

39, 43 $\{(abcd)_8\ (efgh)_8\}_{2,2}\ (ij)$,

44, 50 $\{(abcd)_8\ (efgh)_4\ (ij)\}\ \text{dim.}$

51, 57 $\{(abcd)_8\ (efgh)\ \text{cyc.}\ (ij)\}\ \text{dim.}$

58, 93 $[\{(abcd)_8\ (efgh)_8\}\ \text{dim.}\ (ij)]\ \text{dim.}$

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48. 33, 34 $(abcd.efgh)$ all (ij), $\{(abcd)$ pos. $(efgh)_4(ij)\}$ dim.

35, 36 $\{[(abcd)$ pos. $(efgh)$ cyc.,
$\{(abcd)$ all $(efgh)$ cyc.$\}$ pos.$]$ (ij) dim.

37, 38 $[\{(abcd)$ all $(efgh)_4\}$ dim. $(ij)]$ dim.

64. 31, 32 $(abcd)_8\{(efgh)_4, (efgh)$ cyc.$\}$ (ij),

33, 38 $\{(abcd)_8\ (efgh)_8\}$ dim. (ij),

39, 47 $\{(abcd)_8\ (efgh)_8\ (ij)\}$ dim.

96. 37, 39 $(abcd)$ pos $\{(efgh)_4, (efgh)$ cyc.$\}$ (ij),
$\{(abcd)$ pos. $(efgh)$ pos.$\}_{4,4}$ (ij),

40, 41 $[(abcd)$ all $\{(efgh)_4, (efgh)$ cyc.$\}]$ dim. (ij),

42, 47 $[(abcd)$ all $\{(efgh)_4, (efgh)$ cyc.$\}$ $(ij)]$ dim.

48, 51 $([\{(abcd)$ all $(efgh)$ all$\}_{4,4}$,
$(abcd)$ pos. $(efgh)_8]$ $(ij))$ dim.

52, 60 $[\{(abcd)$ all $(efgh)_8\}$ dim. $(ij)]$ dim.

128. 14 $(abcd)_8\ (efgh)_8\ (ij)$,

192. 25, 27 $(abcd)$ all $\{(efgh)_4, (efgh)$ cyc.$\}$ (ij),
$(abcd)$ pos. $(efgh)_8$ (ij),

28, 31 $\{(abcd)$ all $(efgh)_8\}$ dim. (ij),
$\{(abcd)$ all $(efgh)$ all$\}_{4,4}$ (ij),

32, 38 $\{(abcd)$ all $(efgh)_8\ (ij)\}$ dim.

288. 23, 24 $(abcd)$ pos. $(efgh)$ pos. (ij),
$\{(abcd)$ all $(efgh)$ pos. $(ij)\}$ dim.

25 $[\{(abcd)$ all $(efgh)$ all$\}$ dim. $(ij)]$ dim.

384. 9 $(abcd)_{24}\ (efgh)_8\ (ij)$,

576. 13, 14 $[(abcd)$ all $(efgh)$ pos.,
$\{(abcd)$ all $(efgh)$ all$\}$ dim.$]$ (ij),

15, 16 $\{(abcd)$ all $(efgh)$ all $(ij)\}$ dim.

1152. 7 $(abcd)$ all $(efgh)$ all (ij).

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Systems of intransitivity 5, 3, 2.

30. 1, 2 {(*abcde*) cyc. (*fgh*) cyc. (*ij*)},
{(*abcde*) cyc. (*fgh*) all (*ij*)} dim.

3, 4 {$(abcde)_{10}$ (*fgh*) cyc. (*ij*)} dim.,
[{$(abcde)_{10}$ (*fgh*) all} dim. (*ij*)] dim.

60. 2, 3 (*abcde*) cyc. (*fgh*) all (*ij*),
{$(abcde)_{10}$ (*fgh*) all} dim. (*ij*),

4, 5 $(abcde)_{10}$ (*fgh*) cyc. (*ij*),
{$(abcde)_{20}$ (*fgh*) cyc. (*ij*)} dim.

6, 9 {$(abcde)_{10}$ (*fgh*) all (*ij*)} dim.,
[{$(abcde)_{20}$ (*fgh*) all} pos. (*ij*)] dim.

120. 2, 3 $(abcde)_{10}$ (*fgh*) all (*ij*), $(abcde)_{20}$ (*fgh*) cyc. (*ij*),

4, 7 {$(abcde)_{20}$ (*fgh*) all} pos. (*ij*),
{$(abcde)_{20}$ (*fgh*) all (*ij*)} dim.

240. 5 $(abcde)_{20}$ (*fgh*) all (*ij*),

360. 1, 2 (*abcde*) pos. (*fgh*) cyc. (*ij*),
{(*abcde*) pos. (*fgh*) all (*ij*)} dim.

3, 4 {(*abcde*) all (*fgh*) cyc. (*ij*)} dim.,
[{(*abcde*) all (*fgh*) all} pos (*ij*)] dim.

720. 2, 3 (*abcde*) all (*fgh*) cyc. (*ij*),
(*abcde*) pos. (*fgh*) all (*ij*),

4, 7 {(*abcde*) all (*fgh*) all} pos. (*ij*),
{(*abcde*) all (*fgh*) all (*ij*)} dim.

1440. 8 (*abcde*) all (*fgh*) all (*ij*).

30

Systems of intransitivity 3, 3, 4.

12. 7, 8 (*abc.def*) cyc. {$(ghij)_4$, (*ghij*) cyc.},

9 {(*abc.def*) cyc. (*ghij*) pos.}$_{1,4}$,

10, 11 [(*abc.def*) all {$(ghij)_4$, (*ghij*) cyc.}] dim.

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24. 32, 33 $(abc.def)$ cyc. $(ghij)_8$, $\{(abc.def)$ all $(ghij)$ all$\}_{1,4}$,
34, 35 $(abc.def)$ all $\{(ghij)_4$, $(ghij)$ cyc.$\}$,
36, 38 $\{(abc.def)$ all $(ghij)_8\}$ dim.

36. 5, 6 $\{abc.def)$ cyc. $(ghij)$ pos.,
$\{(abc)$ all (def) all $(ghij)_4\}_{9,1}$,
7, 8 (abc) cyc. (def) cyc. $\{(ghij)_4$, $(ghij)$ cyc.$\}$,
9, 10 $\{(abc)$ cyc. (def) cyc. $(ghij)$ pos.$\}_{3,4}$,
11, 14 $[(abc)$ all (def) cyc., $\{(abc)$ all (def) all$\}$ pos.$]$
$\times[(ghij)_4, (ghij)$ cyc.$]$ dim.

48. 39 $(abc.def)$ all $(ghij)_8$,

72. 24, 25 $(abc.def)$cyc. $(ghij)$ all, $(abc.def)$ all $(ghij)$ pos.
26 $\{(abc.def)$ all $(ghij)$ all$\}$ dim.
27 (abc) cyc. (def) cyc. $(ghij)_8$,
28, 31 $[(abc)$ all (def) cyc., $\{(abc)$ all (def) all$\}$ pos.$]$
$\times[(ghij)_4$, $(ghij)$ cyc.$]$,
32, 37 $([(abc)$ all (def) cyc.,
$\{(abc)$ all (def) all$\}$ pos.$]$ $(ghij)_8)$ dim.
38 $\{(abc)$ all (def) cyc, $(ghij)$ pos.$\}_{6,4}$,
39, 41 $([(abc)$ all (def) cyc.,
$\{(abc)$ all (def) all$\}$ pos.$]$ $(ghij)$ all$)_{3,4}$,
42, 45 $[(abc)$ all (def) all $\{(ghij)_4$, $(ghij)$ cyc.$\}]$ dim.
46, 48 $\{(abc)$ all (def) all $(ghij_8\}_{9,9}$,

108. 1 (abc) cyc. (def) cyc. $(ghij)$ pos.

144. 35, 36 $(abc.def)$ all $(ghij)$ all,
$\{(abc)$ all (def) all $(ghij)$ all$\}_{6,4}$,
37, 38 $[(abc)$ all (def) cyc.,
$\{(abc)$ all (def) all$\}$ pos.$]$ $(ghij)_8$,
39, 40 (abc) all (def) all $\{(ghij)_4$, $(ghij)$ cyc.$\}$,
41, 46 $\{(abc)$ all (def) all $(ghij)_8\}$ dim.

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216. 8 (*abc*) cyc. (*def*) cyc. (*ghij*) all,

4, 5 [(*abc*) all (*def*) cyc.,
{(*abc*) all (*def*) all} pos.] (*ghij*) pos.

6, 7 ([(*abc*) all (*def*) cyc.,
{(*abc*) all (*def*) all} pos.](*ghij*) all) dim.

288. 26 (*abc*) all (*def*) all $(ghij)_8$,

432. 7, 8 [(*abc*) all (*def*) cyc.,
{(*abc*) all (*def*) all} pos.](*ghij*) all,

9, 10 {(*abc*) all (*def*) all (*ghij*) all} dim.

11 (*abc*) all (*def*) all (*ghij*) pos.

864. 7 (*abc*) all (*def*) all (*ghij*) all.

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Systems of intransitivity 6, 2, 2.

6. 1, 2 (*abcdef*) cyc. (*gh.ij*) dim., $(abcdef)_6(gh.ij)$ dim.

12. 12, 18 {(*abcdef*) cyc., $(abcdef)_6$}(*gh.ij*),

14, 17 [{(*abcdef*) cyc., $(abcdef)_6$}{(*gh*)(*ij*)}] dim.

18, 23 $[(abcdef)_{12}\{(gh)(ij)\}]_{3,1}$, $\{(abcdef)_{12}(gh.ij)\}$ dim.

18. 1 $\{(abcdef)_{18}(gh.ij)\}$ dim.

24. 39, 40 {(*abcdef*) cyc., $(abcdef)_6$}(*gh*)(*ij*),

41, 42 $\{(abcdef)_{12}, (abcdef)_{12_2}\}(gh.ij)$,

43, 48 $[(abcdef)_{12}\{(gh)(ij)\}]$ dim.

49, 51 [{(+ *abcdef*)$_{24}$, (± *abcdef*)$_{24}$,
$(abcdef)_{24_5}$}(*gh ij*)] dim.

36. 15, 16 $(abcdef)_{18}(gh.ij)$, $\{(abcdef)_{36}(gh.ij)\}$ dim.

17, 20 $[(abcdef)_{18}\{gh)(ij)\}]$dim., $\{(abcdef)_{36_2}(gh.ij)\}$dim.

21, 22 $[(abcdef)_{36_3}\{(gh)(ij)\}]_{9,1}$,

48. 40, 41 $\{(abcdef)_{12}, (abcdef)_{12_2}\}(gh)(ij)$,

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	42, 44	$\{(+abcdef)_{24}, (\pm abcdef)_{24}, (abcdef)_{24_5}\}(gh.ij)$,
	45, 50	$[\{(+abcdef)_{24}, (\pm abcdef)_{24}, (abcdef)_{24_5}\}\{(gh)(ij)\}]$ dim.
	51, 56	$\{(abcdef)_{48}(gh.ij)\}$ dim., $[(abcdef)_{48}\{(gh)(ij)\}]_{12,1}$,
72.	49, 51	$(abcdef)_{18}(gh)(ij)$, $\{(abcdef)_{36}, (abcdef)_{36_2}\}(gh.ij)$,
	52, 57	$\{(abcdef)_{72}, (gh.ij)\}$ dim., $[(abcdef)_{72}\{(gh)(ij)\}]_{18,1}$,
	58, 63	$[\{(abcdef)_{36}, (abcdef)_{36_2}\}\{(gh)(ij)\}]$ dim.
96.	61, 63	$\{(+abcdef)_{24}, (\pm abcdef)_{24}, (abcdef)_{24_5}\}(gh)(ij)$,
	64, 70	$(abcdef)_{48}(gh.ij)$, $[(abcdef)_{48}\{(gh)(ij)\}]$ dim.
120.	8, 9	$(abcdef)_{60}(gh.ij)$, $\{(abcdef)_{120}(gh.ij)\}$ dim.
144.	47, 49	$\{(abcdef)_{36}, (abcdef)_{36_2}\}(gh)(ij)$, $(abcdef)_{72}(gh.ij)$,
	50, 55	$[(abcdef)_{72}\{(gh)(ij)\}]$ dim.
192.	39	$(abcdef)_{48}(gh)(ij)$,
240.	6, 7	$(abcdef)_{60}(gh)(ij)$, $(abcdef)_{120}(gh.ij)$,
	8, 9	$[(abcdef)_{120}\{(gh)(ij)\}]$ dim.
288.	27	$(abcdef)_{72}(gh)(ij)$,
480.	7	$(abcdef)_{120}(gh)(ij)$,
720.	8, 9	$(abcdef)$ pos. $(gh.ij)$, $(abcdef)$ all $(gh.ij)$ dim.
1440.	9, 10	$(abcdef)$ pos. $(gh)(ij)$, $(abcdef)$ all $(gh.ij)$,
	11, 12	$[(abcdef)$ all $\{(gh)(ij)\}]$ dim.
2880.	8	$(abcdef)$ all $(gh)(ij)$.
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Systems of intransitivity 3, 3, 2, 2.

6.	3, 4	$(abc.def)$ cyc. $(gh)(ij)$ dim.,
		$[\{(abc.def)$ all $(gh)\}$ dim. $(ij)]$ dim.
12.	24, 27	$\{(abc.def)$ all $(gh)(ij)\}$ dim.,
		$(abc.def)$ cyc. $(gh)(ij)$,
18.	2, 3	$\{(abc)$ cyc. (def) cyc. $(gh.ij)$,
		$[\{(abc)$ cyc. (def) all $(gh)\}$ pos. $(ij)]$ dim.
	4	$[\{(abc)$ all (def) all$\}$ pos. $(gh.ij)]$ dim.

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24. 52 (*abc.def*) all (*gh*) (*ij*),

36. 23, 24 (*abc*) cyc. (*def*) cyc. (*gh*) (*ij*),

(*abc*) cyc. (*def*) all (*gh.ij*),

25, 28 ([(*abc*) cyc. (*def*) all,

{(*abc*) all (*def*) all} pos.] {(*gh*) (*ij*)}) dim.

29, 31 {(*abc*) all (*def*) all} pos. (*gh.ij*),

{(*abc*) all (*def*) all (*gh.ij*)} dim.

32, 33 [(*abc*) all (*def*) all {(*gh*) (*ij*)}]$_{3,1}$

72. 64, 65 (*abc*) cyc. (*def*) all (*gh*) (*ij*),

{(*abc*) all (*def*) all} pos. (*gh*) (*ij*),

66, 70 (*abc*) all (*def*) all (*gh.ij*),

[(*abc*) all (*def*) all {(*gh*) (*ij*)}] dim.

144. 56 (*abc*) all (*def*) all (*gh*) (*ij*).

29

Systems of intransitivity 2, 2, 2, 4.

4. 3, 4 [(*ab.cd.ef*) {(*ghij*)$_4$, (*ghij*) cyc.}] dim.

5, 6 ([(*ab.cd*) (*ef*), {(*ab*) (*cd*) (*ef*)} pos.] (*ghij*)$_4$)$_{1,12}$

8. 25, 26 (*ab.cd.ef*) {(*ghij*)$_4$, (*ghij*) cyc.},

27, 29 {(*ab.cd.ef*) (*ghij*)$_8$} dim.

30, 35 [(*ab.cd*) (*ef*) {(*ghij*)$_4$, (*ghij*) cyc.}] dim.

36, 37 [{(*ab*) (*cd*) *ef*)} pos. {*ghij*)$_4$, (*ghij*) cyc.}] dim.

38, 44 ([{(*ab*) (*cd*) (*ef*)} pos., (*ab.cd*) (*ef*)] (*ghij*)$_4$)$_{1,2}$

45, 47 {(*ab*) (*cd*) (*ef*) (*ghij*)$_4$}$_{2,12}$

16. 77, 78 (*ab.cd*) (*ef*) {(*ghij*)$_4$, (*ghij*) cyc.},

79, 80 {(*ab*) (*cd*) (*ef*)} pos. {(*ghij*)$_4$, (*ghij*) cyc.},

81, 92 ([(*ab.cd*) (*ef*), {(*ab*) (*cd*) (*ef*)} pos.] (*ghij*)$_8$) dim.

93, 98 [(*ab*) (*cd*) (*ef*) {(*ghij*)$_4$, (*ghij*) cyc.}] dim.

99, 109 {(*ab*) (*cd*) (*ef*) (*ghij*)$_8$}$_{2,2}$, (*ab.cd.ef*) (*ghij*)$_8$,

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24. 53, 54 $(ab.cd.ef)\ (ghij)$ pos.,
$\{(ab.cd.ef)\ (ghij)$ all$\}$ dim.

32. 94, 95 $(ab)\ (cd)\ (ef)\ \{(ghij)_4,\ (ghij)$ cyc.$\}$,
96, 97 $[(ab.cd)\ (ef),\ \{(ab)\ (cd)\ (ef)\}$ pos.$]\ (ghij)_8$,
98, 106 $\{(ab)\ (cd)\ (ef)\ (ghij)_8\}$ dim.

48. 57, 58 $[(ab.cd)\ (ef),\ \{(ab)\ (cd)\ (ef)\}$ pos.$]\ (ghij)$ pos.
59, 62 $([(ab.cd)\ (ef),$
$\{(ab)\ (cd)\ (ef)\}$ pos.$]\ (ghij)$ all) dim.
68 $(ab.cd.ef)\ (ghij)$ all,

64. 48 $(ab)\ (cd)\ (ef)\ (ghij)_8$,

96. 71, 72 $[(ab.cd)\ (ef),\ \{(ab)\ (cd)\ (ef)\}$ pos.$]\ (ghij)$ all,
73, 75 $\{(ab)\ (cd)\ (ef)\ (ghij)$ all$\}$ dim.
76 $(ab)\ (cd)\ (ef)\ (ghij)$ pos.

192. 40 $(ab)\ (cd)\ (ef)\ (ghij)$ all.
90.

Systems of intransitivity 2, 2, 2, 2, 2.

2. 1 $(ab.cd.ef.gh.ij)$

4. 7, 9 $\{(ab.cd.ef)\ (gh)\ (ij)\}$ dim.
10 $\{(ab.cd)\ (ef.gh)\ (ij)\}$ dim. $s.g = (ab.cd.ef.gh)$,

8. 48, 52 $\{(ab.cd)\ (ef)\ (gh)\ (ij)\}$ dim.
53 $[\{(ab)\ (cd)\ (ef)\ (gh)\}$ pos. $(ij)]$ dim.
$s.g = (ab.cd)\ (ef.gh)$,

16. 110, 113 $\{(ab)\ (cd)\ (ef)\ (gh\ (ij)\}$ dim.

32. 107 $(ab)\ (cd)\ (ef)\ (gh)\ (ij)$.
16.

Total number 994.

University of Michigan,
June, 1894.

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SUR UN MODE DE FORMATION DE CERTAINS GROUPES PRIMITIFS.

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NOUS avons établi* le théorème suivant :

Théorème. De tout groupe simple d'ordre g non premier, on déduit, par la considération de l'isomorphe transitif dont l'ordre égale le degré et de son conjoint, un groupe primitif d'ordre g^2, de degré g, de facteurs de composition g et g.

Nous nous proposons de donner ici en même temps une démonstration un peu différente et une extension de ce théorème, en adoptant un mode de représentation des groupes de substitutions indiqué par M. Cayley.

I.

Soient G un groupe régulier, c'est-à-dire un groupe transitif et dont l'ordre g est égal au degré†; a_1, a_2, a_3, ... a_g ses lettres. Ecrivons sur une même ligne horizontale ces g lettres, dans l'ordre ci-dessus, puis au-dessous et successivement les lettres par lesquelles chaque substitution de G, à part l'unité, remplace respectivement $a_1, a_2, \ldots, a_g$, de façon qu' à chaque substitution corresponde une ligne du tableau ainsi formé; on obtient ainsi un carré de lettres :

$$(1)\quad \begin{cases} a_1^1,\ a_2^1,\ \ldots,\ a_g^1, \\ a_1^2,\ a_2^2,\ \ldots,\ a_g^2, \\ \ldots\ldots\ldots\ldots\ldots, \\ a_1^g,\ a_2^g,\ \ldots,\ a_g^g, \end{cases}$$

tel que $a_1^i, a_2^i, \ldots, a_g^i$ désignent les mêmes g lettres $a_1, a_2, \ldots, a_g$ prises dans un ordre différent pour différentes valeurs de i. Dans ce tableau, l'indice supérieur i distingue les éléments de la $i^{\text{ème}}$ ligne, et l'indice inférieur j les éléments de la $j^{\text{ème}}$ colonne; enfin la substitution

$$S_{ii} = \begin{pmatrix} a_1^1, & a_2^1, & \ldots, & a_g^1 \\ a_1^i, & a_2^i, & \ldots, & a_g^i \end{pmatrix},$$

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* *Thèse de Doctorat*, Paris, Gauthier-Villars, 1892, p. 31.

† Jordan, *Traité des Substitutions*, p. 58 à 60.

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est une substitution de G quel que soit i. Soit $S_{\alpha\beta}$ la substitution qui remplace la $\alpha^{\text{ème}}$ ligne du tableau (1) par la $\beta^{\text{ème}}$ élément à élément; $S_{\alpha\beta} = S_{1\alpha}^{-1} S_{1\beta}$ fait partie de G; de plus en laissant α fixe et faisant varier β on obtient g substitutions évidemment différentes et qui sont dès lors les g substitutions de G: donc on pourra toujours, étant donné α, trouver β tel que la substitution

$$S_{\alpha\beta} = \begin{pmatrix} a_1^{\alpha}, & a_2^{\alpha}, & \ldots, & a_g^{\alpha} \\ a_1^{\beta}, & a_2^{\beta}, & \ldots, & a_g^{\beta} \end{pmatrix},$$

soit une substitution arbitrairement choisie de G.

On sait*, et on voit facilement que chaque substitution de G, à part l'unité, déplace toutes les lettres, et est régulière; il en résulte que les g lettres d'une colonne de (1), $a_j^1, a_j^2, \ldots, a_j^g$ sont différentes, et identiques à l'ordre près à $a_1, a_2, \ldots, a_g$. Le tableau (1) ne comprendra donc pas deux lettres identiques dans une de ses lignes ou une de ses colonnes†.

Réciproquement, considérons a priori un carré analogue à (1) formé avec g lettres $a_1, a_2, \ldots, a_g$ répétées chacune g fois, et assujetti à la seule condition de ne pas comprendre deux lettres identiques dans une de ses lignes ou une de ses colonnes. Le groupe G dérivé de toutes les substitutions

$$S_{\alpha\beta} = \begin{pmatrix} a_1^{\alpha}, & a_2^{\alpha}, & \ldots, & a_g^{\alpha} \\ a_1^{\beta}, & a_2^{\beta}, & \ldots, & a_g^{\beta} \end{pmatrix},$$

sera évidemment transitif; G pourra être un groupe régulier; mais il pourra aussi ne pas l'être, c. à. d. que son ordre sera plus grand que son degré, comme le montre l'exemple du carré formé avec 5 nombres

$$\begin{cases} 1, & 2, & 3, & 4, & 5, \\ 2, & 1, & 4, & 5, & 3, \\ 3, & 4, & 5, & 2, & 1, \\ 4, & 5, & 1, & 3, & 2, \\ 5, & 3, & 2, & 1, & 4, \end{cases}$$

qui conduit à un groupe G de substitutions entre les 5 nombres 1, 2, 3, 4, 5 identique au groupe symétrique de 5 élements.

Or nous pouvons maintenant opérer sur les colonnes du tableau anologue à (1) que nous considérons comme nous l'avons

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* Jordan, *Traité des Substitutions*, p. 58.

† Ce mode de représentation des groupes réguliers a été indiqué, par M. Cayley, *American Journal of Mathematics*, t. I, p. 50 et 174, et t. XI, p. 139.

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fait sur les lignes; nous obtiendrons ainsi un groupe Γ dérivé de toutes les substitutions

$$\Sigma_{\alpha\beta} = \begin{pmatrix} a_\alpha^1, & a_\alpha^2, & \ldots, & a_\alpha^\rho \\ a_\beta^1, & a_\beta^2, & \ldots, & a_\beta^\rho \end{pmatrix}.$$

Γ sera un groupe transitif.

Les groupes G et Γ sont inséparables et inséparables du tableau (1); nous pourrons donc les appeler conjoints. Nous allons voir en effet que, si G est régulier, Γ est formé de l'ensemble des substitutions échangeables à toutes les substitutions de G, par suite aussi régulier; en sorte que, dans ce cas particulier, G et Γ sont ce que M. Jordan a appelé deux groupes conjoints.* Notre définition n'est par suite qu'une extension de celle de M. Jordan.

Soit S une substitution quelconque de G, Σ une substitution échangeable à toutes les substitutions de G, G étant supposé régulier. On a:

$$(2) \quad \Sigma^{-1} S \Sigma = S.$$

Soit $a_{i_1}^\alpha$ une lettre déplacée par Σ. Le tableau (1) ne renfermant pas dans une de ses lignes ou de ses colonnes deux éléments identiques, et, par suite, renfermant toujours dans une quelconque de ses lignes ou de ses colonnes un élément identique à $a_{i_1}^\alpha$, i_1 peut être arbitrairement choisi. Supposons que Σ remplace $a_{i_1}^\alpha$ par a_λ, qui est différent de $a_{i_1}^\alpha$, puisque cette dernière lettre est déplacée par Σ. Il y a toujours dans la $\alpha^{\text{ème}}$ ligne du tableau (1) une lettre $a_{j_1}^\alpha$ identique à a_λ et $j_1 \neq i_1$. S étant quelconque dans G pourra toujours s'écrire

$$S = \begin{pmatrix} a_{i_1}^\alpha, & a_{j_1}^\alpha, & \ldots \\ a_{i_1}^\beta, & a_{j_1}^\beta, & \ldots \end{pmatrix},$$

β étant convenablement choisi. Mais alors l'égalité (2) exige que Σ soit de la forme

$$\Sigma = \begin{pmatrix} a_{i_1}^\alpha, & a_{i_1}^\beta, & \ldots \\ a_{j_1}^\alpha, & a_{j_1}^\beta, & \ldots \end{pmatrix}.$$

En faisant varier S et laissant α et i_1 fixes, par suite aussi j_1, on peut faire prendre à β toutes les valeurs possibles et Σ substitue les éléments de la $j_1^{\text{ème}}$ colonne aux éléments correspondants de la $i_1^{\text{ème}}$, et fait partie de Γ.

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* *Traité des Substitutions*, pp. 58–60.

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$S_{i_i}^2$ sont différentes, et puisque G est régulier, les g valeurs de a_i^i correspondantes doivent être différentes, c.à.d. que la diagonale $a_1^1, \ldots, a_i^i, \ldots, a_g^g$ contient g lettres distinctes qui sont, dans un certain ordre, les lettres $a_1, a_2, \ldots, a_g$.

Si au contraire g est pair, les g substitutions S_{i_i} seront toujours différentes, et on aura autant de valeurs de $S_{i_i}^2$ égales à 1 qu'il y a de substitutions d'ordre 2 dans G. La diagonale $a_1^1, \ldots, a_i^i, \ldots a_g^g$ contiendra donc autant de lettres a_i^i identiques à a_1^1 qu'il y a de substitutions d'ordre 2 dans G.

Donc:—

Si les deux arrangements formés par les lettres de la première ligne et de la première colonne du tableau (1) sont identiques, ce qu'on peut toujours réaliser en permutant les lignes ou les colonnes de ce tableau, et si G est régulier, la condition nécessaire et suffisante pour que les g lettres de la diagonale principale issue du sommet supérieur gauche du tableau soient différentes est que g soit impair.

3°. Le raisonnement fait à la page 121 sur S et Σ, G et Γ est encore applicable quand G n'est pas régulier. On peut raisonner de même pour les substitutions échangeables à toutes les substitutions de Γ: la condition nécessaire pour que S soit échangeable à toutes les substitutions de Γ est qu'on puisse trouver g nombres $\alpha, \beta, \gamma, \ldots$ égaux dans un certain ordre à $1, 2, \ldots, g$ et tels que

$$S = S_{\alpha\beta} = S_{\beta\gamma} = \ldots.$$

Un raisonnement analogue à celui de la page 122 montre ra d'ailleurs que cette condition est suffisante.

Le nombre des substitutions $S_{\alpha'\beta'}$ étant g^2, il y aura au plus g substitutions, toutes contenues dans G, échangeables à toutes les substitutions de Γ; s'il y en a au moins g, G étant dérivé des substitutions $S_{\alpha'\beta'}$, toutes ses substitutions jouiront de la même propriété et devront être de la forme $S_{\alpha'\beta'}$, en sorte que G sera d'ordre g et régulier, ainsi que Γ. S'il y en a moins de g, G étant transitif et d'ordre $\geq g$ contiendra des substitutions non échangeables à toutes celles de Γ; G ne sera pas régulier, pas plus que Γ. Donc:

Le groupe des substitutions échangeables à toutes celles d'un des groupes conjoints dérivé du tableau (1) est contenu dans l'autre conjoint, d'ordre $\leq g$,* et formé de substitutions

* Les substitutions de ce groupe, à part l'unité, déplacent g lettres, puisqu'il en est de même de $S_{\alpha\beta}$ ou $\Sigma_{\alpha\beta}$, et par suite l'ordre de ce groupe est un diviseur de g.

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de la forme $S_{\alpha\beta}$ ou $\Sigma_{\alpha\beta}$. Son ordre ne peut être égal à g que si les conjoints sont réguliers.

4°. Supposons que, par permutation des lignes ou des colonnes, on puisse rendre le tableau (1) symétrique par rapport à une diagonale, et admettons que cette permutation ait été faite. Si cette diagonale est celle qui contient a_g^1, en faisant tourner le tableau de 180° autour d'une colonne, on obtiendra un carré donnant les mêmes groupes G et Γ, et qui, en changeant les indices, prend encore la forme du tableau (1), et est maintenant symétrique par rapport à la diagonale contenant a_1^1 : il suffit donc d'examiner ce dernier cas.

Alors, la condition décessaire et suffisante pour la symétrie en question est $a_i^j = a_j^i$ quels que soient i et j.

Les substitutions

$$S_{\alpha\beta} = \begin{pmatrix} a_{i_1}{}^{\alpha}, & a_{j_1}{}^{\alpha}, & a_{k_1}{}^{\alpha}, & \dots \\ a_{i_1}{}^{\beta}, & a_{j_1}{}^{\beta}, & a_{k_1}{}^{\beta}, & \dots \end{pmatrix},$$

et

$$\Sigma_{\alpha\beta} = \begin{pmatrix} a_{\alpha}{}^{i_1}, & a^{j_1}{}_{\alpha}, & a^{k_1}{}_{\beta}, & \dots \\ a^{i_1}{}_{\beta}, & a^{j_1}{}_{\beta}, & a^{k_1}{}_{\beta}, & \dots \end{pmatrix},$$

sont identiques. Les deux conjoints G et Γ sont identiques, et, par suite, si G est régulier, il est formé de substitutions échangeables. De plus, d'après ce qu'on a vu tout-à-l'heure, les substitutions échangeables à toutes les substitutions de G seront contenues dans G. Donc :

Si par permutation des lignes ou des colonnes le tableau (1) peut être rendu symétrique par rapport à une diagonale, les deux conjoints sont identiques : l'un des conjoints contient toutes les substitutions qui sont échangeables à toutes ses substitutions, et, s'il est régulier, il est formé de substitutions échangeables.

Etablissons la réciproque pour les conjoints réguliers. La condition nécessaire et suffisante pour que deux conjoints G et Γ réguliers soient identiques est que le groupe G soit formé de substitutions échangeables. Supposons donc que ceci ait lieu.

Dans le carré (1) qui correspond à G permutons par exemple les colonnes de façon que les arrangements formés par les lettres de la première ligne et de la première colonne du nouveau tableau (1) obtenu soient identiques, c. à. d. que

$$a_1^i = a_i^1.$$

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On aura

$$S_{1\beta} = \begin{pmatrix} a_{i_1}^{\ 1}, & a_{j_1}^{\ 1}, & a_{k_1}^{\ 1}, & \dots \\ a_{i_1}^{\ \beta}, & a_{j_1}^{\ \beta}, & a_{k_1}^{\ \beta}, & \dots \end{pmatrix},$$

$$\Sigma_{1\beta} = \begin{pmatrix} a_1^{\ i_1}, & a_1^{\ j_1}, & a_1^{\ k_1}, & \dots \\ a_\beta^{\ i_1}, & a_\beta^{\ j_1}, & a_\beta^{\ k_1}, & \dots \end{pmatrix};$$

$S_{1\beta}$ remplace $a_1^{\ 1}$ par $a_1^{\ \beta}$ et $\Sigma_{1\beta}$ remplace $a_1^{\ 1}$ par $a_\beta^{\ 1} = a_1^{\ \beta}$; G et Γ étant identiques et réguliers, G ne contient qu'une substitution remplaçant $a_1^{\ 1}$ par $a_1^{\ \beta}$, et

$$S_{1\beta} = \Sigma_{1\beta}.$$

Dès lors, à cause de $a_{j_1}^{\ 1} = a_1^{\ j_1}$, quel que soit j_1, il faudra $a_{j_1}^{\ \beta} = a_\beta^{\ j_1}$; β étant d'ailleurs quelconque, le carré (1) sera symétrique par rapport à la diagonale $a_1^{\ 1}, \dots, a_g^{\ g}$.

La condition nécessaire et suffisante pour qu'un groupe régulier coïncide avec son conjoint (ou soit formé de substitutions échangeables) est que la tableau (1) qui lui correspond puisse, par une permutation de ses lignes ou de ses colonnes, être rendu symétrique par rapport à une diagonale.

III.

Etudions maintenant le groupe $H = (G, \Gamma)$ dérivé de deux conjoints G et Γ: H est transitif, cherchons à quelles conditions il est primitif.

Supposons que H ne soit pas primitif; s'il admet une répartition de ses lettres en systèmes de non-primitivité* de n lettres et si $g = mn$, cette répartition sera évidemment commune à G et Γ. On pourra toujours par une permutation convenable des colonnes du tableau (1) faire en sorte que les systèmes soient

$$(3) \quad a^1_{\ 1}, a^1_{\ 2}, \dots a^1_{\ n}, \dots a^1_{\ \phi_{n+1}} \dots a^1_{\ (\phi+1)_n} \dots\dots$$

Si ensuite on permute convenablement les lignes du tableau (1), on pourra faire en sorte que $a_1^{\ \alpha} = a_\alpha^{\ 1}$ et écrire les systèmes (3) sous la forme

$$(4) \quad a_1^{\ 1}, a_1^{\ 2}, \dots a_1^{\ n}, \dots a_1^{\ \phi^{n+1}}, \dots a_1^{\ (\phi+1)^n}; \dots\dots$$

Ceci posé, soit

$$S_{1\alpha} = \begin{pmatrix} a_1^{\ 1} & a_2^{\ 1} & \dots & a_g^{\ 1} \\ a_1^{\ \alpha} & a_2^{\ \alpha} & \dots & a_g^{\ \alpha} \end{pmatrix},$$

* Jordan, *Traité des Substitutions*, p. 34.

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α étant quelconque ; si $\phi < \frac{\alpha}{n} \leq \phi + 1$, les n lettres $a_1^{\alpha}, \ldots a_n^{\alpha}$ coïncideront à l'ordre près avec celles du $(\phi+1)^{\text{ème}}$ systéme (3) ou (4), puisque a_1^{α} en fait partie. Il en sera de même pour les n lettres $a_{\alpha}^1, \ldots, a_{\alpha}^n$.

Plus généralement, considérons le carré

$$(5)\begin{cases} a^{\phi n+1}_{\psi n+1}, & a^{\phi n+1}_{\psi n+2}, & \ldots\ldots\ldots & a^{\phi n+1}_{(\psi+1)n}, \\ a^{\phi n+2}_{\psi n+1}, & a^{\phi n+2}_{\psi n+2}, & \ldots\ldots\ldots & a^{\phi n+2}_{(\psi+1)n}, \\ \ldots\ldots\ldots\

de même de Γ et de H; G, Γ et H sont de plus composés avec des groupes intransitifs.

Théorème. Le groupe dérivé de deux conjoints ne peut être non-primitif que s'il est composé, et que si chacun des deux conjoints est composé avec un groupe intransitif.

Corollaire. Si l'un des conjoints est simple, le groupe dérivé des deux conjoints est primitif.

Désignons le carré (5) par $A_{\psi+1}^{\phi+1}$. La disposition des carrés $A_{\psi+1}^{\phi+1}$ dans le tableau (1) est la suivante :

$$(6)\quad \begin{cases} A_1^1 & A_2^1 \ldots\ldots A_m^1 \\ A_1^2 & A_2^2 \ldots\ldots A_m^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_1^m & A_2^m \ldots\ldots A_m^m \end{cases},$$

Nous avons vu qu'au carré (5), par suite à $A_{\psi+1}^{\phi+1}$, correspondait un seul des systèmes (3) ou (4). Si l'on regarde comme identiques les symboles $A_{\psi+1}^{\phi+1}$ qui correspondent à un même système, le carré (6) est analogue au carré (1): on peut lui faire correspondre deux conjoints G' et Γ', l'un dérivé des substitutions entre les lignes de (6) élément à élément, l'autre des substitutions entre les colonnes.

Alors, soit

$$S_{\alpha\beta} = \begin{pmatrix} a_1^\alpha & a_2^\alpha & \ldots & a_\varrho^\alpha \\ a_1^\beta & a_2^\beta & \ldots & a_\varrho^\beta \end{pmatrix},$$

une substitution de G. Les lettres a_1^α, a_2^α, ..., a_ϱ^α font partie d'un certain nombre de carrés $A_{\psi+1}^{\phi+1}$ compris dans une même ligne de (6); de même pour a_1^β, a_2^β, ..., a_ϱ^β. De plus les lettres a_i^β et a_i^α dont la première est substituée à la deuxième par $S_{\alpha\beta}$ font partie de deux carrés $A_{\psi+1}^{\phi+1}$ situés dans la même colonne de (6). Il en résulte qu'à la substitution $S_{\alpha\beta}$ de G correspond une substitution $S_{\alpha'\beta'}$ de G', et une seule; la substitution $S_{\alpha'\beta'}$ peut d'ailleurs être regardée comme représentant une substitution entre les systèmes (3) et (4), et cette substitution sera précisément celle qu'opére $S_{\alpha\beta}$ entre ces systèmes.

G' étant dérivé des substitutions $S_{\alpha'\beta'}$ et G des substitutions $S_{\alpha\beta}$, G' sera le groupe dérivé des substitutions opérées par G entre les systèmes; demême pour Γ' et Γ.

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Le groupe $H' = (G', \Gamma')$ sera donc le groupe des substitutions opérées par H entre les systèmes (3) et (4). On voit facilement que la condition nécessaire et suffisante pour que H soit primitif est que la répartition (3) soit maxima parmi les répartitions communes à G et Γ.

Remarque. Le carré (6), où l'on regarde comme identiques les symboles $A_{\psi+1}^{\phi+1}$ correspondant à un même système (3) ou (4) donne un carré analogue au carré (1). Chacun des carrés $A_{\psi+1}^{\phi+1}$ est d'ailleurs analogue au carré (1).

Inversement, considérons un carré (6) formé avec m éléments $a_1, a_2, \ldots, a_m$ répétés chacun m fois, et ne contenant pas dans une quelconque de ses lignes ou de ses colonnes deux éléments identiques. Aux m éléments identiques à a_i du carré (6) faisons correspondre m carrés formés chacun avec les mêmes n éléments $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$, chacun de ces n carrés ne renfermant pas deux éléments identiques dans une quelconque de ses lignes ou de ses colonnes, et supposons que les mn éléments $\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1, \alpha_1^2, \ldots, \alpha_n^m$ soient distincts. En disposant dans le carré (6) à la place des m éléments identiques à a_i les m carrés de n éléments $\alpha_1^i, \ldots, \alpha_n^i$ correspondants, on obtient évidemment un carré jouissant de propriétés analogues à celles de (1) c. à. d. formé avec mn éléments, et ne renfermant pas deux éléments identiques dans une ligne ou une colonne quelconque. Les deux conjoints correspondants seront évidemment composés chacun avec un groupe intransitif, et admettrons une répartition commune de leurs lettres n à n, $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$ formant un système de la répartition quel que soit i.

Cas où les conjoints sont réguliers. Ce qui précède est applicable aux conjoints réguliers; mais, à leur égard, on peut aller un peu plus loin, et démontrer que si l'un des conjoints est régulier et composé, il en est de même de l'autre.

Soit G un des conjoints qui est régulier et composé avec un groupe K d'ordre n. On sait que Γ est régulier; je dis que Γ est aussi composé avec un groupe d'ordre n.

En effet, G étant régulier, chacune de ses substitutions (à part l'unité) déplace toutes les lettres régulièrement, et il en est de même pour celles de K. Le groupe K permutera donc transitivement les lettres de G n à n, et puisque K est permutable aux substitutions de G, G admettra une répartition de ses lettres en systèmes de non-primitivité de n lettres*,

* Jordan, *Traité des Substitutions*, p. 41.

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K effectuant d'ailleurs entre les n lettres de chaque système les substitutions d'un groupe régulier.

On pourra disposer les lignes et les colonnes du tableau (1) correspondant à G de façon que les systèmes soient encore les systèmes (3) et (4), et qu'on ait de plus $a_1^i = a_i^1$. Considérons un carré (5). L'ensemble des lettres d'un système (3) étant remplacé exclusivement par l'ensemble des lettres d'une ligne de (5) dans une substitution $S_{1\alpha}$ de G, avec $\phi < \frac{\alpha}{n} \leq \phi + 1$, chaque ligne de (5) est un des systèmes (3); en particulier, cela a lieu pour $\phi = 0$. Mais, de plus, K permute transitivement les lettres de chaque système (3) et contient toute substitution de G qui remplace une lettre d'un système (3) par une lettre d'un même système, sans quoi G contiendrait deux substitutions remplaçant une lettre par une autre lettre et ne pourrait être régulier comme on le voit facilement. Dès lors, puisque, d' après (4), les substitutions $S_{\alpha\beta}$, avec

$$(7) \quad \phi < \frac{\alpha}{n} - \phi + 1, \qquad \phi < \frac{\beta}{n} \leq \phi + 1,$$

simultanément, remplacent une lettre a_1^α d'un système par une lettre a_1^β d'un même système, ces substitutions $S_{\alpha\beta}$ font partie de K; toutes les substitutions $S_{\alpha\beta}$ satisfaisant aux conditions (7) font partie de K et permutent exclusivement entre elles les lettres de chaque système (3); d'autre part elles substituent les lettres d'une ligne de (5) qui forment un système (3) aux lettres d'une autre ligne de (5) qui forment aussi un système (3); ces deux derniers systèmes doivent donc coïncider et les lettres de (5) font toutes partie d'un même système (3).

Il en résulte immédiatement que Γ admet aussi une répartition de ses lettres n à n, répartition qui lui est commune avec G et n'est autre que (3) ou (4). En raisonnant comme nous l'avons fait au début de ce paragraphe dans le cas des conjoints ayant une répartition commune, on voit que Γ est composé avec un groupe intransitif; on pourra choisir d'ailleurs ce groupe comme dérivé des substitutions $\Sigma_{\alpha'\beta'}$ avec $\alpha' \leq n$, $\beta' \leq n$, car ce qui précède montre de suite que ces substitutions permuteront exclusivement entre elles les lettres de chaque système (4). Γ étant régulier, et la groupe dérivé des substitutions $\Sigma_{\alpha'\beta'}$ que nous venons de fixer étant transitif entre les lettres de chaque système, ce dernier groupe sera d'ordre n, et Γ sera comme G composé avec un groupe intransitif d'ordre n.

On peut donc dire:

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Théorème. Si l'un des conjoints est régulier et composé avec un groupe d'ordre n, il en est de même de l'autre. Si l'un des conjoints est régulier et simple, il en est de même de l'autre.

Supposons toujours les deux conjoints réguliers: s'ils ont quelque substitution S commune on peut toujours la choisir d'ordre premier. S faisant partie de l'un des conjoints Γ est échangeable à toutes les substitutions de l'autre G; S faisant partie de G est échangeable à toutes les substitutions de Γ. Le groupe des puissances de S est donc permutable aux substitutions des groupes G et Γ qui sont composés: la seule exception est le cas où le groupe des puissances de S coïncide avec G, par suite avec Γ, puisqu'il est commun à G et Γ et que G et Γ sont de même ordre.

Ainsi, à part le cas où G et Γ sont d'ordre premier, ils ne peuvent avoir quelque substitution commune que s'ils sont composés.

Le groupe H dérivé de G et Γ est d'ailleurs ici, ainsi qu'on le verrait facilement,* d'ordre $\frac{g^2}{L}$, L étant l'ordre du groupe des substitutions communes à G et Γ.

Quand donc G et Γ sont simples, à part le cas où ils sont d'ordre premier, le groupe $H=(G, \Gamma)$, qui est alors primitif, d'après ce qui précède, a pour facteurs de composition† g et g, et est d'ordre g^2. D'où ce théorème:

Théorème.‡ Le groupe dérivé d'un groupe régulier simple et de son conjoint est primitif, d'ordre g^2, de facteurs de composition g et g, sauf quand g est premier.

D'ailleurs, on sait, d'après M. Jordan, qu'à tout groupe d'ordre g on peut faire correspondre un groupe régulier d'ordre g qui lui est holoédriquement isomorphe.§ Le théorème précédent donne donc:

Théorème.‖ A tout groupe simple d'ordre g non premier on peut faire correspondre un groupe primitif d'ordre g^2, de degré g, de facteurs de composition g et g.

Cas où les deux conjoints sont identiques. Comme nous l'avons vu ce sera en particulier le cas si le tableau (1) est symétrique par rapport à une diagonale; alors $H=(G, \Gamma)=G$. Nous avons vu que H ne peut être non-primitif que si G est

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* Serret, *Algèbre Supérieure*, t. II., p. 283; voir aussi notre Thèse de Doctorat, p. 104.

† Jordan, *Traité des Substitutions*, p. 41.

‡ Voir notre Thèse de Doctorat, p. 81; Théorème XIV.

§ Jordan, *Traité des Substitutions*, p. 56.

‖ Voir notre Thèse de Doctorat, p. 81; Théorème XV.

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composé, avec un groupe intransitif et que si G est simple, H est primitif. D'ailleurs si G est composé avec un groupe intransitif, il ne pourra être primitif.* Donc:

Théorème. Quand les deux conjoints sont identiques à un même groupe G, ce qui a lieu en particulier quand le tableau (1) est symétrique par rapport à une diagonale, la condition nécessaire et suffisante pour que G soit primitif est qu'il ne soit pas composé avec un groupe intransitif.

Toulouse, le 12 Mai, 1894.

APPLICATION DE LA THÉORIE DES SUBSTITUTIONS À CELLE DES CARRÉS MAGIQUES.†

Par EDMOND MAILLET, Ingénieur des Ponts et Chaussées, Rue des Jardins 3, Toulouse.

SOIT un carré A, de côté n, formé avec les n^2 premiers nombres $0, 1, 2, \ldots, n^2-1$; les n^2 premiers nombres sont tous de la forme $a+\alpha n$, où a et α prennent toutes les valeurs $0, 1, 2, \ldots, n-1$.

A ce carré on pourra faire correspondre deux carrés B et B', l'un formé avec les quantités a, l'autre avec les quantités α, et tel que les quantités a et α faisant partie dans B et B' respectivement de la $i^{\text{ème}}$ ligne et de la $j^{\text{ème}}$ colonne soient celles qui entrent dans le nombre $a+\alpha n$ faisant partie dans A de la $i^{\text{ème}}$ ligne et de la $j^{\text{ème}}$ colonne.

Considérons par exemple le carré B.

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* Jordan, *Traité des Substitutions*, p. 41.

† Soit un carré de côté n formé avec n^2 nombres consécutifs

$$\begin{cases} a', b', c', \ldots, l', \\ a'', b'', c'', \ldots, l'', \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(n)}, b^{(n)}, c^{(n)}, \ldots, l^{(n)}. \end{cases}$$

Si dans chaque ligne et chaque colonne la somme des n nombres qui y sont contenus est égale à une même quantité q, le carré est dit semi-magique; si de plus la somme des n nombres contenus dans chaque diagonale principale est égale à q, le carré est dit magique; si de plus la somme des n nombres contenus dans chaque diagonale complète est égale à q, le carré est dit diabolique.

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$$(1)\quad B=\begin{cases} a_0^0\ ,\ a_1^0\ ,\ \ldots,\ a_{n-1}^0 \\ a_0^1\ ,\ a_1^1\ ,\ \ldots,\ a_{n-1}^1 \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ a_0^{n-1},\ a_1^{n-1},\ \ldots,\ a_{n-1}^{n-1}, \end{cases}$$

Il pourra se faire que, pour toute ligne ou pour toute colonne, les nombres a_i^j soient tous différents; de même pour le tableau B'. Cela aura lieu par exemple quand on part du carré.

$$A_1=\begin{cases} 23,\ 11,\ 4,\ 17,\ 5, \\ 19,\ 7,\ 20,\ 13,\ 1, \\ 10,\ 3,\ 16,\ 9,\ 22, \\ 6,\ 24,\ 12,\ 0,\ 18, \\ 2,\ 15,\ 8,\ 21,\ 14, \end{cases}$$

qui donne les carrés élémentaires,

$$B_1=\begin{cases} 3,\ 1,\ 4,\ 2,\ 0, \\ 4,\ 2,\ 0,\ 3,\ 1, \\ 0,\ 3,\ 1,\ 4,\ 2, \\ 1,\ 4,\ 2,\ 0,\ 3, \\ 2,\ 0,\ 3,\ 1,\ 4, \end{cases} \qquad B_1'=\begin{cases} 4,\ 2,\ 0,\ 3,\ 1, \\ 3,\ 1,\ 4,\ 2,\ 0, \\ 2,\ 0,\ 3,\ 1,\ 4, \\ 1,\ 4,\ 2,\ 0,\ 3, \\ 0,\ 3,\ 1,\ 4,\ 2. \end{cases}$$

Nous considèrerons exclusivement les carrés A conduisant à des carrés B et B' jouissant chacun de cette propriété au moins pour toute ligne ou pour toute colonne.

Supposons par exemple que dans B elle ait lieu pour toutes les lignes; écrivons l'un au-dessous de l'autre les éléments de la $(1+j)^{\text{ème}}$ ligne et de la $(1+k)^{\text{ème}}$ ligne de B, ainsi qu'il suit :

$$(2)\quad \begin{pmatrix} a_0^j, & a_1^j, & \ldots, & a_{n-1}^j \\ a_0^k, & a_1^k, & \ldots, & a_{n-1}^k \end{pmatrix}.$$

On sait que la notation (2) représente une substitution†; désignons-la par S_{jk}. En faisant varier j et k de toutes les manières possibles, on obtiendra un certain nombre de

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* La considération des carrés élémentaires B et B' serait due à La Hire. Voir à ce sujet, Le problème d'Euler et les carrés magiques, de M. Frolow, p. 30 (Saint-Pétersbourg, imprimerie Trenké et Fusnot, 1884).

† Voir par exemple Serret, *Algèbre Supérieure*, 1889, t. II, p. 244.

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substitutions : le groupe G dérivé de ces substitutions les contiendra toutes et pourra être regardé comme correspondant à B.

Il pourra d'ailleurs se faire que la propriété qu'on a supposée avoir lieu pour les lignes de B ait lieu aussi pour les colonnes. C'est par exemple le cas pour B_1 et B_1'. Alors, en opérant sur les colonnes de B comme on l'a fait sur les lignes, on pourre faire correspondre à B un second groupe G_1 dérivé des substitutions

$$(3) \quad \Sigma_{il} = \begin{pmatrix} a_i^0, & a_i^1, & \ldots, & a_i^n \\ a_l^0, & a_l^1, & \ldots, & a_l^n \end{pmatrix},$$

i et l prenant toutes les valeurs $0, 1, 2, \ldots, n-1$. Dans ce cas on voit de suite que B ne contiendra pas deux éléments identiques dans une ligne quelconque ou dans une colonne quelconque.

Réciproquement, soit G un groupe dérivé de n substitutions entre n lettres ou nombres $0, 1, 2, \ldots, n-1$. Chacune de ces n substitutions pourra toujours se mettre sous la forme S_{0k}, k prenant successivement les valeurs $0, 1, 2, \ldots, n-1$, selon que l'on considère la première, la $2^{\text{ème}}$, ..., la $n^{\text{ème}}$ de ces n substitutions ; $a_0^0, a_1^0, \ldots, a_{n-1}^0$ d'une part, $a_0^k, a_1^k, \ldots, a_{n-1}^k$ d'autre part, désignant n éléments ou nombres $0, 1, 2, \ldots, n-1$ dans un ordre différent ou non.

On voit immédiatement* qu'aux n substitutions S_{0k}, ou au groupe G, on pourra faire correspondre un tableau B analogue à (1), où la $(1+k)^{\text{ème}}$ ligne sera formée des éléments ou nombres que S_{0k} substitue respectivement à $a_0^0, a_1^0, \ldots, a_{n-1}^0$. Ce tableau B ne renfermera pas deux éléments identiques dans une même ligne, d'après la définition des substitutions, et jouira par suite des propriétés supposées aux carrés B et B' que nous avons considérés. Enfin le groupe correspondant à B se confondra avec G, car il est dérivé des substitutions S_{jk}, par suite dest substitutions S_{0k}, puisque l'on évidemment

$$S_{jk} = S_{0j}^{-1} S_{0k}.$$

En opérant sur un second groupe G', analogue à G, et différent on non de G, on pourra former un second tableau B' analogue à B.

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* Ce mode de représentation des groupes de substitution, qui peut ne pas convenir à tous les groupes, a été indiqué, pour les groupes transitifs dont l'ordre égale le degré (groupes réguliers), par M. Cayley (*American Journal of Mathematics*, tome I, p. 50 et 174. et tome XI, p. 139). 'Certains groupes peuvent aussi être représentés par des rectangles, des parallélipipédes rectangles, des cubes de nombre et peut-être aussi par des figures analogues dans des espaces à plusde 8 dimensions'

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Si alors nous formons un carré A tel que l'élément de A appartenant à la $(1+j)^{ème}$ ligne et à la $(1+i)^{ème}$ colonne soit $a_i^j + a_i^j n$, le carré A sera formé avec n^2 nombres inférieurs à n^2. Il y aura à examiner si ces n^2 nombres sont distincts et si le carré A jouit de certaines propriétés, comme par exemple d'être magique ou semi-magique. La condition nécessaire et suffisante pour que le carré A soit formé avec n^2 nombres distincts est évidemment que, b étant un quelconque des nombres 0, 1, 2, ..., $n-1$, aux n valeurs des a_i^j, égales à b, correspondent n valeurs des a_i^j distinctes, ou que à un alignement formé dans B par les n nombres égaux à b corresponde dans B' un alignement de n nombres distincts.

Plus généralement, on pourra opérer sur un transformé quelconque de G, et sur un de G', par des substitutions entre les nombres 0, 1, 2, ..., $n-1$, comme on l'aura fait sur G et G'. On remarquera d'ailleurs que la carré analogue à B correspondant au transformé $U^{-1}GU$ de G par une substitution U s'obtiendra simplement en effectuant dans B la substitution U.

On remarquera également que, dans le carré B, on pourra, sans changer G, permuter d'une façon quelconque les lignes et les colonnes; de même pour B'; puis opérer encore sur les nouveaux carrés obtenus.

En résumé, dans certains cas, on pourra faire correspondre à un carré formé avec les n^2 premiers nombres et de côté n au moins deux groupes de substitutions, différents ou non, entre n lettres ou nombres.

Réciproquement, dans certains cas, on pourra faire correspondre à deux groupes de substitutions, différents ou non, entre n lettres, et dérivés chacun de n substitutions, un carré de côté n formé avec les n^2 premiers nombres.

On pourra donc se proposer les deux problèmes suivants :

1°. Etant donnés un carré de côté n formé avec les n^2 nombres 0, 1, 2, ..., n^2-1, trouver, s'ils existent, les groupes de substitutions correspondants.

Réciproquement :

2°. Etant donnés deux groupes de degré n dérivés chacun de n substitutions, trouver, s'ils existent, les carrés de côté n formés avec les n^2 nombres 0, 1, 2, ..., n^2-1, et correspondants.

On pourra évidemment, dans le cas du second problème, restreindre l'étendue de la question en supposant que les carrés

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cherchés jouissent de certaines propriétés, comme d'être semi-magiques, magiques ou diaboliques.*

Carrés formés avec les n^2 premiers nombres.

La condition nécessaire et suffisante pour que la carré A dérivé de B et de B' soit formé avec les n^2 premiers nombres est qu'à tout alignement formé dans B par n nombres égaux corresponde dans B' un alignement formé avec les n nombres inégaux $0, 1, 2, \ldots, (n-1)$ dans un ordre quelconque.

Soit alors

$$B = \begin{cases} a_0^0, a_1^0, \ldots, a_{n-1}^0, \\ a_0^1, a_1^1, \ldots, a_{n-1}^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ a_0^{n-1}, a_1^{n-1}, \ldots, a_{n-1}^{n-1}. \end{cases}$$

Considérons un carré B_0 dont chaque ligne soit constituée par les n nombres égaux à une des quantités $0, 1, 2, (n-1)$ faisant partie de B, et soit

$$B_0 = \begin{cases} b_0^0, b_1^0, \ldots, b_{n-1}^0, \\ b_0^1, b_1^1, \ldots, b_{n-1}^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ b_0^{n-1}, b_1^{n-1}, \ldots, b_{n-1}^{n-1}, \end{cases}$$

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* A un carré magique de côté n ne correspond pas toujours un groupe, au moins avec la définition que nous avons donnée des groupes correspondants, définition qui est vraisemblablement susceptible d'extension. Ainsi le carré magique

$$A_2 = \begin{cases} 6, 21, 24, 7, 2, \\ 5, 11, 16, 9, 19, \\ 4, 10, 12, 14, 20, \\ 23, 15, 8, 13, 1, \\ 22, 3, 0, 17, 18, \end{cases}$$

qui conduit aux carrés

$$B_2 = \begin{cases} 1, 1, 4, 2, 2, \\ 0, 1, 1, 4, 4, \\ 4, 0, 2, 4, 0, \\ 3, 0, 3, 3, 1, \\ 2, 3, 0, 2, 3, \end{cases} \qquad B_2' = \begin{cases} 1, 4, 4, 1, 0, \\ 1, 2, 3, 1, 3, \\ 0, 2, 2, 2, 4, \\ 4, 3, 1, 2, 0, \\ 4, 0, 0, 3, 3, \end{cases}$$

n'a pas de groupe correspondant.

Un carré B de la forme

$$\begin{cases} 0, 1, 0, 1, \\ 1, 0, 1, 0, \\ 0, 1, 0, 1, \\ 1, 0, 1, 0, \end{cases}$$

a aussi pour groupe correspondant le groupe dérivé des puissances de la substitution $(0, 1)$. Nous ne nous occupon pas ici des extensions semblables de la définition des groupes correspondants.

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avec $b_0^i = b_1^i = \ldots = b_{n-1}^i$, chaque colonne de B_0 contenant n nombres distincts.

Soit un carré

$$C = \begin{cases} c_0^0 , c_1^0 , \ldots, c_{n-1}^0, \\ c_0^1 , c_1^1 , \ldots, c_{n-1}^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ c_0^{n-1}, c_1^{n-1}, \ldots, c_{n-1}^{n-1}, \end{cases}$$

ayant dans chacune de ses lignes n éléments distincts et formé avec les n nombres $0, 1, 2, \ldots, n-1$ répétés n fois (on obtient par exemple un carré de ce genre en faisant tourner le carré B de 90° ou en prenant $C = B$, quand B n'a pas deux éléments identiques dans une quelconque de ses lignes ou de ses colonnes).

Formons un carré

$$D = \begin{cases} d_0^0 , d_1^0 , \ldots, d_{n-1}^0, \\ d_0^1 , d_1^1 , \ldots, d_{n-1}^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ d_0^{n-1}, d_1^{n-1}, \ldots, d_{n-1}^{n-1}, \end{cases}$$

en substituant dans B à l'élément a_k^i représenté dans B_0 par b_i^j l'élément c_i^j du carré C. On voit de suite qu'à tout alignement de B formé de n nombres égaux à b_i^j correspondra dans D un alignement formé avec les n nombres d'une ligne de C, c'est-à-dire n nombres différents. Donc, en prenant $D = B'$, le carré A dérivé sera formé avec les n^2 premiers nombres.

On pourra évidemment supposer qu'on permute préalablement dans B_0 les lignes, les colonnes, ou, dans chaque ligne de B_0 les élements d'une manière quelconques.

On voit ainsi que D possède un système de n alignements distincts, n'ayant deux à deux aucun élément* commun, qui ne se confondent pas avec les lignes ou les colonnes, formés chacun de n nombres distincts et tels que les n alignements correspondants dans B soient formés chacun de nombres égaux.

Réciproquement soit D un carré possédant n alignements distincts, n'ayant deux à deux aucun emplacement commun et formés chacun de n nombres distincts. Si on substitue aux

* L élément peut être considéré au point de vue de sa valeur numérique, ou de son emplacement. Il a pu nous arriver de prendre le mot élément dans l'un ou l'autre sens. Nous ne croyons pas qu'il puisse en résulter grande confusion.

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éléments de ces n alignements n nombres égaux à un seul des nombres 0, 1, 2, ..., $(n-1)$ pour chaque alignement, de façon que les nombres d'un alignement ne soient pas répétés dans un autre, on obtiendra un carré B tel que le carré A dérivé de B et de $B'=D$ soit formé avec les n^2 premiers nombres.

Carrés semi-magiques.

Supposons en particulier que le carré D dont nous venons de partir jouisse des propriétés suivantes :

1°. Il n'a pas deux éléments identiques dans une ligne ou dans une colonne quelconque.

2°. Les n alignements distincts peuvent être choisis de façon qu'aucun d'eux n' ait deux éléments communs avec une ligne ou une colonne quelconque de D.

Il en résulte que chaque alignement, qui contient n éléments, aura un élément et un seul commun avec une ligne ou une colonne quelconque de D.

On aura donc dans chaque ligne ou chaque colonne de B un élément et un seul qui soit égal à un des nombres 0, 1, 2, ..., $n-1$ arbitrairement choisi, et B n'aura pas non plus deux éléments identiques dans une ligne ou une colonne quelconque.

B jouira également d'une propriété semblable à la seconde des propriétés que nous avons supposée à D. On le voit facilement en remarquant que les n alignements distincts de B seront les alignements de B correspondant aux n alignements de D formés chacun par n nombres égaux.

B et D jouissent donc de propriétés correspondantes.

Considérons maintenant le carré A dérivé de B et de $B'=D$, et formons par exemple la somme des nombres contenus dans une ligne de A. Cette somme sera $\Sigma_B+n\Sigma_{B'}$, Σ_B et $\Sigma_{B'}$ désignant la somme des éléments contenus dans la ligne correspondante des carrés B et B' respectivement.

Or chacune de ces lignes renferme dans un certain ordre les n éléments distincts 0, 1, 2, ..., $n-1$. Donc

$$\Sigma_B=\frac{n(n-1)}{2},\quad \Sigma_{B'}=\frac{n(n-1)}{2},\quad \Sigma_B+n\Sigma_{B'}=\frac{(n+1)\,n(n-1)}{2}.$$

De même pour une colonne quelconque ; il en résulte que la somme des nombres de toute ligne ou de toute colonne de A est une même quantité, et par suite que le carré A est semi-magique. Donc :

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Théorème.—Soit un carré B' de côté n formé avec les n nombres 0, 1, 2, ..., $n-1$ répétés chacun n fois, et qui ne renferme pas dans une ligne ou une colonne quelconque deux éléments identiques. Si on peut trouver dans B' n alignements formés chacun de n nombres distincts, n'ayant deux à deux aucun emplacement commun; et n'ayant avec chaque ligne ou chaque colonne qu'un emplacement commun; si alors on substitue aux nombres de chacun de ces alignements n nombres identiques à l'un des nombres 0, 1, 2, ..., $n-1$, de façon que les nombres substitués soient distincts d'un alignement à l'autre, on obtient un carré B tel que le carré A dérivé de B et de B' est semi-magique.

On peut d'ailleurs montrer qu'il existe des carrés B' jouissant des propriétés supposées.

Considérons un groupe régulier G de substitutions, c'est-à-dire un groupe transitif et dont l'ordre égale le degré. Supposons que ce degré soit un nombre n impair, et représentons les éléments par 0, 1, 2, ..., $(n-1)$.

Nous pouvons représenter ce groupe par un carré n'ayant pas deux éléments identiques dans une ligne ou une colonne quelconque, et qu'on pourra écrire :

$$D=\begin{cases} d_0^0\ ,\ d_1^0\ ,\ \ldots,\ d_{n-1}^0, \\ d_0^1\ ,\ d_1^1\ ,\ \ldots,\ d_{n-1}^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ d_0^{n-1},\ d_1^{n-1},\ \ldots,\ d_{n-1}^{n-1}. \end{cases}$$

Nous pourrons supposer qu'on y ait ordonné les lignes ou les colonnes de façon que $d_i^0=d_0^i$ quel que soit i. Je dis qu'alors les n éléments $d_0^0, d_1^1, \ldots, d_{n-1}^{n-1}$ sont distincts.

En effet, supposons qu'on ait $d_i^i=d_j^j$. Soit

$$S_{0i}=\begin{pmatrix} d_0^0, & d_i^0, & \ldots \\ d_0^i, & d_i^i, & \ldots \end{pmatrix},$$

on aura :

$$S^n{}_{0i}=\begin{pmatrix} d_0^0, & \ldots \\ d_i^i, & \ldots \end{pmatrix},$$

puisque $d_0^i=d_i^0$.

De même

$$S^n{}_{0j}=\begin{pmatrix} d_0^0, & \ldots \\ d_j^j, & \ldots \end{pmatrix},$$

j étant différent de i.

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G étant régulier ne peut renfermer deux substitutions distinctes remplaçant un même élément d_0^0 par un même élément $d_i^i = d_j^j$. Donc

$$S_{0i}^{\ 2} = S_{0j}^{\ 2}.$$

Soient λ_i, λ_j les ordres respectifs de S_{0i}, S_{0j}, qui sont impairs puisque n est impair. On peut trouver u_i tel que $2u_i \equiv 1$, (mod. λ_i); alors:

$$(S^2_{0i})^{u_i} = S_{0i} = S_{0j}^{\ 2u_i} = (S^2_{0i})^{u_i+\lambda_i} = S_{0j}^{\ 2u_i+2\lambda_i},$$

ce qui donne $S_{0j}^{\ 2\lambda_i} = 1$, en sorte que λ_j étant impair divise λ_i; on verrait de même que λ_i doit diviser λ_j et par suite que $\lambda_i = \lambda_j$, ce qui exige,

$$S_{0i} = S_{0j}^{\ 2u_i} = S_{0j},$$

alors qu'évidemment S_{0i} diffère de S_{0j} puisque ces deux substitutions remplacent un même nombre d_0^0 par des nombres d_0^i et d_0^j différents. L'hypothèse $d_i^i = d_j^j$ avec $i \neq j$ conduit donc à une contradiction et les n éléments de la diagonale $d_0^0, d_1^1, \ldots, d_{n-1}^{n-1}$ sont distincts.

Dans le carré D, amenons une colonne autre que la première à la place de la première, la deuxième colonne par exemple. Puis permutons les colonnes autres que la nouvelle première colonne ainsi obtenue et les lignes, de façon que le nouveau carré

$$E = \begin{cases} e_0^{\ 0}, e_1^{\ 0}, \ldots, e_{n-1}^{0}, \\ e_0^{\ 1}, e_1^{\ 1}, \ldots, e_{n-1}^{1}, \\ \ldots\ldots\ldots\ldots\ldots, \\ e_0^{n-1}, e_1^{n-1}, \ldots, e_{n-1}^{n-1}, \end{cases}$$

obtenu par ce moyen soit tel que $e_i^{\ 0} = e_0^{\ i} = d_i^{\ 0} = d_0^{\ i}$ quel que soit i.

A cause de l'égalité $e_i^{\ 0} = e_0^{\ i}$ quel que soit i, la diagonale $e_0^{\ 0}, e_1^{\ 1}, \ldots, e_{n-1}^{n-1}$ du nouveau carré contient encore n éléments distincts. Les éléments de cette diagonale occupaient certaines places dans le carré D; je dis qu'aucun d'eux ne se trouvait dans la diagonale $d_0^0, d_1^1, \ldots, d_{n-1}^{n-1}$.

En effet, soit

$$T_{0i} = \begin{pmatrix} e_0^{\ 0}, & e_i^{\ 0}, & \ldots \\ e_0^{\ i}, & e_i^{\ i}, & \ldots \end{pmatrix}.$$

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On a $e_0^0 = d_0^0$, $e_i^0 = e_0^i = d_i^0 = d_0^i$, en sorte que $S_{0i} = T_{0i}$, le groupe G étant régulier, et que $d_i^j = e_i^i$. Dès lors les deux diagonales considérées renfermant chacune n éléments distincts, le seul élément de la seconde qui pouvait occuper dans la première la place d_i^i doit être égal à e_i^i, et par suite est e_i^i.

Or soit d_k^k l'emplacement qu'occupait e_0^0 dans D. On a $k \neq 0$ et $k \neq 1$, puisque $e_0^0 = d_0^0$ et que $d_0^0 \neq d_1^0$ et $d_0^0 \neq d_k^k$.

e_i^i se trouvait dans D dans la même colonne que e_i^0, qui est égal à d_i^0, et d_i^i dans la même colonne que d_i^0; mais e_i^0 était dans D dans la même ligne que e_0^0, par suite dans la $(k+1)^{\text{ème}}$ ligne, avec $k > 1$, c'est-à-dire dans une ligne différente de celle où était d_i^0; par conséquent, à cause $d_i^0 = e_i^0$, e_i^0 ne se trouvait pas non plus dans la même colonne que d_i^0, sans quoi une colonne de D aurait renfermé deux éléments identiques, contrairement à ce qui a lieu. On en conclut que e_i^i ne se trouvait pas dans D dans la même colonne que d_i^i, et, a fortiori, que e_i^i n'occupait pas dans D la place d_i^i.

Il en résulte, puisque i est quelconque, qu'aucun des éléments de la diagonale $e_0^0, e_1^1 \ldots e_{n-1}^{n-1}$ de E n'occupait dans D un emplacement de la diagonale $d_0^0, d_1^1 \ldots d_{n-1}^{n-1}$.

En considérant la 3ème colonne de D et opérant sur elle comme nous l'avons fait sur la 2ème, nous trouverions un carré F anologue à D et E et dont la diagonale $f_0^0, f_1^1 \ldots f_{n-1}^{n-1}$ jouirait par rapport aux diagonales correspondantes de D et E de propriétés analogues à celles que nous venons d'établir; et ainsi de suite.

On obtient donc n carrés D, E, F, ... dont les diagonales $d_0^0, d_1^1 \ldots d_{n-1}^{n-1}$, ... renferment chacune n éléments distincts. Considérons les n alignements formés dans D par les emplacements qu'occupent primitivement dans D les éléments de ces n diagonales; supposons que ces alignements ne soient pas distincts et que deux d'entre eux, ceux correspondant à E et F par exemple, aient un emplacement commun. Pour former F, on peut évidemment déduire E de D, puis F de E; dans ces transformations, deux alignements ayant un emplacement commun se transforment en deux alignements ayant un emplacement commun; d'après l'hypothèse que nous avons faite les éléments de la diagonale de $F, f_0^0, f_1^1, \ldots, f_{n-1}^{n-1}$ occuperaient dans E un alignement ayant avec la diagonale $e_0^0, e_1^1, \ldots, e_{n-1}^{n-1}$ un emplacement commun, ce qui n'a pas lieu, puisqu'on peut raisonner sur E et F comme nous l'avons fait sur D et E. Donc les n alignements de D que nous avons considérés n'ont

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deux à deux aucun élément commun ; chacun d'eux est d'ailleurs formé de n nombres distincts et n'a avec une ligne ou une colonne quelconque de D qu'un emplacement commun, puisqu'il en est ainsi par exemple pour l'alignement $f_0^0, f_1^1, \ldots, f_{n-1}^{n-1}$ de F dans le carré F et que les éléments d'une ligne ou d'une colonne de F occupent dans D une même ligne ou une même colonne.

Le carré D remplit donc les conditions prévues au théorème précédent pour le carré B' et on en conclut :

Théorème. A tout groupe régulier de substitutions d'ordre n impair on peut faire correspondre au moins un carré semi-magique, de côté n.

On sait qu'à tout groupe de substitutions correspond un groupe régulier de même ordre* ; donc

Théorème. A tout groupe de substitutions d'ordre n impair on peut faire correspondre au moins un carré semi-magique de côté n.

Dans ce dernier théorème la notion de groupe correspondant doit évidemment être conçue dans un sens plus général que d'après notre définition.

Carré magique déduit du groupe des puissances de la substitution (*abcde*).

$5a+2$	$5b+a'$	$5c+b'$	$5d+c'$	$10+d'$
$5b+d'$	$5c+2$	$5d+a'$	$10+b'$	$5a+c'$
$5c+c'$	$5d+d'$	12	$5a+a'$	$5b+b'$
$5d+b'$	$10+c'$	$5a+d'$	$5b+2$	$5c+a'$
$10+a'$	$5a+b'$	$5b+c'$	$5c+d'$	$5d+2$

Dans ce carré on peut donner à a, b, c, d dans un ordre quelconque les valeurs 0, 1, 3, 4 ; de même pour a', b', c', d'. Le carré obtenu est magique.

* Jordan, *Traité des Substitutions*, p. 58.

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Carré magique déduit du groupe transitif d'ordre et de degré 9 formé des puissances de la substitution (*abcdefghk*).

$9a+4$	$9b+a'$	$9c+b'$	$9d+c'$	$9e+d'$	$2f+e'$	$9g+f'$	$9h+g'$	$36+h'$
$9b+h'$	$9c+4$	$9d+a'$	$9e+b'$	$9f+c'$	$9g+d'$	$9h+e'$	$36+f'$	$9a+g'$
$9c+g'$	$9d+h'$	$9e+4$	$9f+a'$	$9g+b'$	$9h+c'$	$36+d'$	$9a+e'$	$9b+f'$
$9d+f'$	$9e+g'$	$9f+h'$	$9g+4$	$9h+a'$	$36+b'$	$9a+c'$	$9b+d'$	$9c+e'$
$9e+e'$	$9f+f'$	$9g+g'$	$9h+h'$	40	$9a+a'$	$9b+b'$	$9c+c'$	$9d+d'$
$9f+d'$	$9g+e'$	$9h+f'$	$36+g'$	$9a+h'$	$9b+4$	$9c+a'$	$9d+b'$	$9e+c'$
$9g+c'$	$9h+d'$	$36+e'$	$9a+f'$	$9b+g'$	$9c+h'$	$9d+4$	$9e+a'$	$9f+b'$
$9h+b'$	$36+c'$	$9a+d'$	$9b+e'$	$9c+f'$	$9d+g'$	$9e+h'$	$9f+4$	$9g+a'$
$36+a'$	$9a+b'$	$9b+c'$	$9c+d'$	$9d+e'$	$9e+f'$	$9f+g'$	$9g+h'$	$9h+4$

Dans ce carré on peut donner à a, b, c, d, e, f, g, h dans un ordre quelconque les valeurs 0, 1, 2, 3, 5, 6, 7, 8; de même pour $a', b', c', d', e', f', g', h'$; le carré obtenu est toujours magique.

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Carré magique déduit du groupe transitif d'ordre et de degré 9 dérivé des substitutions $(abc)\,(def)\,(ghk)$ et de $(adg)\,(beh)\,(cfk)$.

$9a+4$	$9b+h'$	$9c+g'$	$9d+f''$	$9e+e'$	$9f+d''$	$9g+c'$	$9h+b'$	$36+a'$
$9b+g'$	$9c+4$	$9a+h'$	$9e+d''$	$9f+f''$	$9d+e'$	$9h+a'$	$36+c'$	$9g+b'$
$9c+h'$	$9a+g'$	$9b+4$	$9f+e'$	$9d+d''$	$9e+f''$	$36+b'$	$9g+a'$	$9h+c'$
$9d+c'$	$9e+b'$	$9f+a'$	$9g+4$	$9h+h'$	$36+g'$	$9a+f''$	$9b+e'$	$9c+d''$
$9e+a'$	$9f+c'$	$9d+b'$	$9h+g'$	40	$9g+h'$	$9b+d''$	$9c+f''$	$9a+e'$
$9f+b'$	$9d+a'$	$9e+c'$	$36+h'$	$9g+g'$	$9h+4$	$9c+e'$	$9a+d''$	$9b+f''$
$9g+f''$	$9h+e'$	$36+d''$	$9a+c'$	$9b+b'$	$9c+a'$	$9d+4$	$9e+h'$	$9f+g'$
$9h+d''$	$36+f'$	$9g+e'$	$9b+a'$	$9c+c'$	$9a+b'$	$9e+g'$	$9f+4$	$9d+h'$
$36+e'$	$9g+d''$	$9h+f''$	$9c+b'$	$9a+a'$	$9b+c'$	$9f+h'$	$9d+g'$	$9e+4$

Dans ce carré on peut donner à a, b, c, d, e, f, g, h dans un ordre quelconque les valeurs 0, 1, 2, 3, 5, 6, 7, 8; de même pour $a', b', c', d', e', f', g', h'$: le carré obtenu est toujours magique.

Toulouse, le 12 *Mai*, 1894.

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NOTE ON GEODESIC TORSION.

By *G. B. Mathews.*

LET P be any point of a curve drawn on a given surface: PT, PN, PB the tangent, principal normal, and binormal of the curve at P; PG the normal to the surface at P (fig. 36).

Let $P'T'$, $P'N'$, $P'B'$, $P'G'$ be the corresponding lines for a consecutive point P' on the curve.

Let $BPG = \theta$, $B'P'G' = \theta + d\theta$.

Then if $PP' = ds$, the system of axes PT, PN, PB can be brought into the position $P'T'$, $P'N'$, $P'B'$ by means of a translation $(ds, 0, 0)$ followed by a rotation $(-\tau ds, 0, \kappa ds)$, τ, κ being the tortuosity and curvature at P.

The line PG, supposed rigidly connected with the moving axes, will be brought into the position $P'G_1$, coplanar with $P'N'$, $P'B'$ and such that $B'P'G_1 = \theta$.

Hence a further rotation $(-d\theta, 0, 0)$ will bring $P'G_1$ into the position $P'G'$.

On the whole, then, the transition from PG to $P'G'$ is effected by a movement compounded of the translation

$$(ds, 0, 0)$$

and the rotation

$$(-\tau ds - d\theta, 0, \kappa ds).$$

Now by the definition of the geodesic tortuosity γ, the first component of the rotation last written is $-\gamma ds$, with a convention as to the sense in which γ is to be reckoned positive: therefore

$$\gamma ds = \tau ds + d\theta,$$

or

$$\gamma = \tau + \frac{d\theta}{ds}.$$

This is the formula obtained by a much more laborious process in Laurent's *Traité d'Analyse*, t. VII., p. 25.

When the curve is a line of curvature $\gamma = 0$, and consequently

$$\tau + \frac{d\theta}{ds} = 0,$$

which is Lancret's theorem.

In order to realise more thoroughly the meaning of this result, consider the osculating plane at P, and the normal plane to the surface which contains PT. Let the point P move along the curve, PT moving with it so as always

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to touch the curve, and let the two planes through PT rotate about it in such a way as always to remain the osculating plane of the curve at P and the normal plane to the surface specified above. Then $-\tau$ measures the rate (per unit of arc) at which the osculating plane turns round PT, and $-\frac{d\theta}{ds}$ the rate at which the normal plane turns round. If, then, a plane be drawn through PT bisecting the angle between the other two planes, the rate at which the new plane turns round is one half of the geodesic tortuosity. The rotation is, of course, relative to fixed axes with which the moving axes for the moment coincide.

Most of the corollaries given by Laurent follow very easily; there is one, due to Serret, which admits of a simpler demonstration than that given in the treatise. The theorem is this:

If a line of curvature, C, drawn on a given surface is plane, then the cuspidal edge of the developable generated by the normals to the surface at points on the line of curvature is a helix.

To prove this we observe that since $\tau = 0$ and $\gamma = 0$ it follows that $\frac{d\theta}{ds} = 0$: hence the normals are all inclined at the same angle to the line which is perpendicular to the plane of C. Call this angle α. Now take a sphere, centre O and radius unity, and draw radii of this sphere parallel to the generating lines of the developable. The extremities of these lines will trace out a small circle on the sphere of angular radius α. If radii of the sphere are drawn parallel to the binormals of C their extremities lie on another small circle parallel to the former and of angular radius $\frac{1}{2}\pi - \alpha$. Let OT, OT', OB, OB' be radii corresponding to two consecutive tangents and two consecutive binormals. Then if κ, τ are the curvature and tortuosity at the corresponding point on the cuspidal edge,

$$\kappa : \tau = \text{angle } TOT' : \text{angle } BOB'.$$

Now the spherical arcs TB, $T'B'$ cut the small circles at right angles: hence

$$\text{angle } TOT' : \text{angle } BOB' = \sin\alpha : \sin(\tfrac{1}{2}\pi - \alpha),$$

and therefore

$$\frac{\kappa}{\tau} = \tan\alpha,$$

a constant; and this proves the theorem.

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ON THE REDUCTION OF KRONECKER'S MODULAR SYSTEMS.

By HARRIS HANCOCK.

Introduction.

THE discovery of the modular systems by Professor Kronecker is one of the greatest steps that has been made towards the advancement from the usual theory of numbers to the arithmetical treatment of integral functions with indeterminate coefficients.

By means of these systems the theory of higher congruences is set forth in a new light, and many fields belonging to the theory of numbers which have hitherto been widely separated may, by means of them, be brought into closer connection. For example, a close intimacy is revealed between the theory of algebraic forms and that of the ideal numbers, and in either case the advantage of much greater generality is derived.

On the one hand the theory of integral algebraic forms may be subjected not only to multiplication but also to addition, subtraction and division in the same way that integral algebraic functions are treated; and, on the other hand, the theory of ideal numbers as well as the theory of complex systems of numbers, which before the introduction of modular systems had been treated only under limiting conditions, may, by the means of these systems, be treated in a complete and simple manner.

The reduction of complex numbers into their ideal prime factors is nothing else than the reduction of modular systems into their irreducible factors, so that the abstract properties of ideal numbers take now a concrete form. We have also a higher plane for the consideration of geometrical questions; for, consider a modular system consisting of three variables, and let these three variables form a system of coordinates in space. Then, surfaces may be represented by modular systems of the first kind, curves by modular systems of the second kind, and systems of points by modular systems of the third kind.

The notion of modular systems is first given in Kronecker's *Festschrift zu Herrn Kummer's Doctorjubiläum*, which is reprinted in Crelle's *Journal*, Vol. XCII., pp. 1—123.

In this memoir applications are made to purely arithmetical

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questions; and again (*Crelle*, Vol. XCIX., pp. 329—371) Prof. Kronecker makes beautiful applications of modular systems to elementary algebraic questions, deducing results in very precise and more general forms.

I have made great use of these two papers in the preparation of the following article, in fact, the first four or five pages of my work, which contain the principal definitions and the conception of the theory of modular systems, have been taken almost directly from these papers.

I have also used freely notes on the subject which I had the privilege of taking from Professor Kronecker during the Winter Semester 1891—92, and in a conversation with him was suggested the problem of the reduction of those modular systems, the results of which are given Arts. 36 et seq.

In closing I wish to express many thanks to Prof. Hensel of the Berlin University for reading the manuscript and making many suggestions.

1. Gauss (Art. 1, *Disq. Arith.*) makes the following definition: if the difference of the numbers a and b is divisible by m, then a and b are said to be *congruent* with respect to m; if not, incongruent; we call m the *modulus*, and each of the numbers a and b in the first case is said to be a *residue* of the other, and in the second case a *non-residue*.

And again (Art. 2) all residues of a given number a with respect to a modulus m are embraced under the form

$$a + km,$$

where k is an indeterminate integer.

Hence the *totality* of all integers that are expressible in the form

$$\ldots a - 3m,\ a - 2m,\ a - m,\ a,\ a + m,\ a + 2m,\ a + 3m,\ \ldots$$

is denoted by saying that they are congruent to a, *modulo* m, and the series of numbers $a + km$ is completely determined through two quantities, viz. the fixed modulus m and any other *term* of the series.

2. Following Kronecker ('Crelle,' Vol. 99, pp. 330 et seq.), we define two linear forms of any variables x and x',

$$a + bx,\ a' + b'x',$$

as *equivalent* to each other, if the one can be transferred into the other by an integral substitution

$$x = \alpha x' + \beta,\ x' = \alpha' x + \beta'.$$

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The necessary and sufficient conditions for this equivalence are therefore

$$b = \pm b',\ a \equiv a' \text{ (mod. } b),$$

so that the conception of the congruence of the numbers $a \equiv a'$ (mod. m) is quite the same as the conception of the equivalence of the linear forms $a + mx \backsim a' + mx'$.

3. The natural extension of the conception of the "congruence with regard to one modulus" to the more general conception of the "congruence with regard to a system of moduli" is at once suggested, when, instead of the linear forms with one variable, we consider linear forms with any number of variables

$$a + m_1x_1 + m_2x_2 + \ldots + m_\mu x_\mu;$$

and, as above, we define two forms,

$$a + \sum_{h=1}^{h=\mu} m_h x_h,\quad a' + \sum_{\kappa=1}^{\kappa=\nu} m'_\kappa x'_\kappa,$$

as equivalent to each other, when the one can be transferred into the other by the integral substitutions

$$x_h = c_{h0} + \sum_\kappa c_{h\kappa} x'_\kappa,\ x'_\kappa = c'_{\kappa 0} + \sum_h c'_{\kappa h} x_h,$$

$$\begin{pmatrix} h = 1, 2, \ldots, \mu \\ \kappa = 1, 2, \ldots, \nu \end{pmatrix}.$$

Hence, the necessary and sufficient conditions for the equivalence of the forms

$$a + \sum_{h=1}^{h=\mu} m_h x_h,\quad a' + \sum_{\kappa=1}^{\kappa=\nu} m'_\kappa x'_\kappa$$

are expressed by the equations

$$(A.)\quad a = a' + \sum_\kappa c'_{\kappa 0} m'_\kappa,\quad a' = a + \sum_h c_{h0} m_h$$

$$(B.)\quad m_h = \sum_\kappa c'_{\kappa h} m'_\kappa,\qquad m'_\kappa = \sum_h c_{h\kappa} m_h$$

$$\begin{pmatrix} h = 1, 2, \ldots, \mu \\ \kappa = 1, 2, \ldots, \nu \end{pmatrix}.$$

The integral coefficients c and c' in equations $(A.)$ and $(B.)$ have special values. Let them have any integral values;

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then, as in the case of one modulus, the "totality" of the numbers that are expressible in the form

$$a + \sum_{h=1}^{h=\mu} c_{h0} m_h$$

is denoted by saying that they are congruent to each other with respect to the modular system $(m_1, m_2, \ldots, m_\mu)$.

For brevity, we call this last modular system M. And any quantity t belonging to the form

$$a + \sum_{h=1}^{h=\mu} c_{h0} m_h$$

we write in the form

$$t \equiv a \text{ (modd. } m_1, m_2, \ldots, m_\mu),$$

or briefly

$$t \equiv a \text{ (mod. } M).$$

In other words t is expressible by means of the quantity a and linear terms in the m's with integral coefficients; and, when a does not occur in the form, t is a linear homogeneous function of the m's with integral coefficients, and may be written

$$t \equiv 0 \text{ (mod. } M).$$

4. If we take into consideration the relations $(B.)$ above, the totality of the numbers which come under the two forms

$$a + \sum_{h=1}^{h=\mu} c_{h0} m_h, \quad a' + \sum_{\kappa=1}^{\kappa=\nu} c'_{\kappa 0} m'_\kappa,$$

is the same, so that the conception of the equivalence of the two modular systems

$$(m_1, m_2, \ldots, m_\mu), \ (m_1', m_2', \ldots, m_\nu')$$

is a natural consequence. These equations $(B.)$ are therefore known as the *characteristic* for the equivalence

$$(m_1, m_2, \ldots, m_\mu) \backsim (m_1', m_2', \ldots, m_\nu').$$

Hence, the necessary and sufficient conditions for the equivalence of the linear forms

$$a + \sum_{h=1}^{h=\mu} m_h x_h, \quad a' + \sum_{\kappa=1}^{\kappa=\nu} m_\kappa' x_\kappa'$$

are expressed by the congruence

$$a \equiv a' \text{ (modd. } m_1, m_2, \ldots, m_\mu),$$

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together with the equivalence

$$(m_1, m_2, \ldots, m_\mu) \backsim (m_1', m_2', \ldots, m_\nu').$$

5. Let M' denote the modular system $(m_1', m_2', \ldots, m_\nu')$, and suppose that each of the elements m' is congruent to zero, modulo M; then, if

$$a \equiv a' \text{ (mod. } M'),$$

it follows that

$$a \equiv a' \text{ (mod. } M).$$

For, from the definition,

$$a - a' = g_1 m_1' + g_2 m_2' + \ldots + g_\nu m_\nu',$$

where the g's are integers; and, since each m' is a linear homogeneous function with integral coefficients of the m's, it is clear that

$$a - a' = g_1 h_1 m_1 + g_2 h_2 m_2 + \ldots + g_\nu h_\mu m_\mu,$$

where the h's are also integers; *i.e.*

$$a - a' \equiv 0 \text{ (mod. } M),$$

or

$$a \equiv a' \text{ (mod. } M).$$

In this case the modular system M' is said to *contain* the modular system M. Hence, from our notion of equivalence above, two modular systems M and M' are equivalent, when each element of either system contains the other system, or, what amounts to the same thing, when every element of one system can be expressed as a linear homogeneous function with integral coefficients of the elements of the other system, and *vice versa*.

Again, if

$$a \equiv 0 \text{ (modd. } M', M),$$

and if each element of M' *contains* M, *i.e.*, if it is congruent to zero, modulo M, then M' can be omitted from the above system, leaving simply

$$a \equiv 0 \text{ (mod. } M).$$

For

$$a \equiv 0 \text{ (mod. } M)$$

means that a is a linear homogeneous function of the m's, and since the m''s are linear homogeneous functions of the m's, the system M' can always be formed when we have the system M, so that M' in the modulus gives us nothing new.

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Hence *any element may be added to a modular system or taken away from it, when this element is a linear homogeneous function of the remaining elements of the modular system.*

6. The question naturally arises how can we reduce modular systems to their simplest forms?

Let us first consider a system $(m_1, m_2, \dots m_\mu)$, where the m's are all integers, and let $m_1 > m_2 > m > \dots > m_\mu$.

Suppose that

$$\frac{m_1}{m_2} = q + \frac{m_1'}{m_2}, \text{ where } q \text{ is an integer,}$$

so that $$m_1' = m_1 - qm_2.$$

This m_1' is a linear homogeneous function of m_1 and m_2, and besides contains the G.C.D. of m_1 and m_2 as a factor.

Hence, instead of having m_1 in our modular system, we may have another equivalent modular system in which m_1' takes the place of m_1, this m_1' being a smaller integer than m_1, *i.e.*,

$$(m_1, m_2, \dots m_\mu) \sim (m_1', m_2, \dots m_\mu);$$

and continuing this process, we finally have

$$(m_1, m_2, \dots m_\mu) \sim (t),$$

where t is the G.C.D. of the elements that compose the system.

7. Let R', R'', ..., $R^{(n)}$ be n indeterminate variables, indeterminate in the sense that at any moment we can give them determinate values. All rational functions of the R's constitute what may be called the "*realm of rationality*" (Rationalitäts-Bereich) $(R', R'', \dots, R^{(n)})$. The realm of rationality contains therefore all those quantities which are rational functions of the quantities $R', R'', \dots, R^{(n)}$ with integral coefficients. If the variables $R', R'', \dots$ are independent, the realm of rationality is said to be *natural*; and when only integral functions of the R's with integral coefficients appear we have an "integral-realm."

The realm of integral numbers is thus extended to the one which embraces the conception of all integral functions with integral coefficients, and that these functions are susceptible to an arithmetical treatment lies in the fact that every integral function with integral coefficients has the fundamental property in common with integers, viz., the unique multiplicative decomposition into prime quantities.

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8. Proceeding further we may extend the conception of congruences (Arts. 3 and 4) to the more general case where $A, M_1, M_2, \dots M_\mu, A_1', M_1', M_2', \dots M_\nu'$ are integral quantities of the realm of rationality $(R', R'', \dots R^{(n)})$. We define the equivalence of two modular systems

$$(M_1, M_2, \dots M_\mu), \ (M_1', M_2', \dots M_\nu'),$$

and the congruence of two quantities A, A' with regard to one of these modular systems, by a system of equations

$$(A). \quad A = A' + \sum_\kappa C_{\kappa_0}' M_\kappa', \quad A' = A + \sum_h C_{h_0} M_h,$$

$$(B). \quad M_h = \sum_\kappa C_{\kappa h}' M_\kappa', \quad M_\kappa' = \sum_h C_{h\kappa} M_h,$$

$$\begin{pmatrix} h = 1, 2, \dots \mu \\ \kappa = 1, 2, \dots \nu \end{pmatrix}$$

in which the C's and C''s are likewise integral quantities of the realm of rationality $(R', R'', \dots R^{(n)})$.

These same equations serve also to define the equivalence of the linear forms

$$A + \sum_{h=1}^{h=\mu} M_h X_h, \quad A' + \sum_{\kappa=1}^{\kappa=\nu} M'_\kappa X'_\kappa,$$

since by means of equations (A) and (B) these forms may be transformed the one into the other by means of substitutions with integral coefficients which belong to the realm of rationality $(R', R'', \dots R^{(n)})$; and the equivalence of these two linear forms is in its turn characterized by the congruence

$$A \equiv A' (\text{modd. } M_1, M_2, \dots M_\mu),$$

in connection with the equivalence

$$(M_1, M_2, \dots M_\mu) \sim (M_1', M_2', \dots M_\nu').$$

Again, if the μ congruences

$$M_h \equiv 0, \ (\text{modd. } M_1', M_2', \dots, M_\nu'), \qquad (h = 1, 2, \dots, \mu)$$

exist, the modular system $(M_1, M_2, \dots, M_\mu)$ *contains* the system $(M_1', M_2', \dots, M_\nu')$, and if the systems mutually contain each other we have the equivalence

$$(M_1, M_2, \dots, M_\mu) \sim (M_1', M_2', \dots, M_\nu').^*$$

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* Cf. Kronecker, *Crelle*, Vol. XCIX., pp. 332, et seq., and also *Crelle*, Vol. XCII., pp. 77, et seq.

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in both the original systems is contained also in the third system; for it is clear that if $(N_1, N_2, \ldots, N_\lambda)$ is contained in both $(M_1, M_2, \ldots, M_\mu)$ and $(M_1', M_2', \ldots, M_\nu')$, then it is contained in $(M_1, M_2, \ldots, M_\mu, M_1', M_2', \ldots, M_\nu')$, since $(M_1, M_2, \ldots, M_\mu, M_1', M_2', \ldots, M_\nu')$ is expressible linearly in terms of $(N_1, N_2, \ldots, N_\lambda)$.

12. *Multiplication and Division.* If $(M_1, M_2, \ldots, M_\mu)$ denotes one modular system and $(M_1', M_2', \ldots, M_\nu')$ another, then the multiplication (or composition) of these two systems gives rise to $\mu.\nu$ elements, each of these being the product of any term in the first system by any term in the second system, and the composed system may be said to contain the other two systems as *factors* or *divisors*.

13. When a modular system is not decomposable into other modular systems (in the sense of equivalence), it is known as an *irreducible* modular system. Irreducible modular systems that do not contain other modular systems of the same kind (cf. Art. 24) are called *prime* modular systems, and when the G. C. D. of two systems is equivalent to *unity*, the two systems are *relative prime* to each other.

14. If (M') is one modular system and (M'') another modular system, and if

$$(M') \sim (M''),$$

it is clear that

$$(M)(M') \sim (M)(M''),$$

where (M) denotes any other modular system.

The linear form of the product of two modular systems is the product of the linear forms of the two systems and can be reduced from them by means of a bilinear substitution.

15. In the case of common numbers we know that if

$$m'.m'' \equiv 0 \pmod{m},$$

then

$$m'' \equiv 0 \left(\text{mod. } \frac{m}{(m, m')}\right).^*$$

* We use the notation (m, m') to denote the G. C. D. of the two numbers m and m'.

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For multiplying both sides of this congruence by (m, m'), we have

$$(m, m')\, m'' \equiv 0 \pmod{m},$$

or
$$(mm'', m'm'') \equiv 0 \pmod{m},$$

therefore
$$m'.m'' \equiv 0 \pmod{m},$$

Again,
$$m \equiv 0 \left(\text{mod. } \frac{m'.m''}{(m', m'')}\right),$$

where $\frac{m'.m''}{(m', m'')}$ is the L. C. M. of m' and m''.

In precisely the same manner, if F, G' and G'' denote modular systems, and if $F \equiv 0$ (modd. G', G''), then

$$F \equiv 0 \left(\text{mod. } \frac{G'.G''}{(G', G'')}\right).$$

16. If the product of two modular systems is divisible by a third modular system which is a prime system, and if the first system is relative prime to the third system, then the second system must be divisible by the third.

For if
$$(M')(M'') \equiv 0 \pmod{M},$$

then
$$(M', M)(M'') \equiv 0 \pmod{M}.$$

But by hypothesis $(M', M) \sim 1$,

therefore
$$(M'') \equiv 0 \pmod{M}.$$

A more extended proof of this proposition will be given later (Art. 20).

17. *A generalization of Fermat's theorem.*

If $x_1, x_2, \ldots, x_\nu$ be any ν variables, then it is clear that

$$(x_1 + x_2 + \ldots + x_\nu)^p \equiv x_1^p + x_2^p + \ldots + x_\nu^p \pmod{p},$$

where p denotes any prime number, *i.e.*,

$$\left(\sum_{h=1}^{h=\nu} x_h\right)^p \equiv \sum_{h=1}^{h=\nu} x_h^p \pmod{p}.$$

Also
$$\left(\sum_{h=1}^{h=\nu} x_h^p\right)^p \equiv \sum_{h=1}^{h=\nu} x_h^{p^2} \pmod{p};$$

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and, *in general*,

$$\left(\sum_{h=1}^{h=\nu} x_h\right)^{p^r} \equiv \sum_{h=1}^{h=\nu} x_h^{p^r} \pmod{p},$$

$$\text{or } \left(\sum_{h=1}^{h=\nu} x_h\right)^{p^r} \equiv \sum_{h=1}^{h=\nu} (x_h^{p^r} - x_h) + \sum_{h=1}^{h=\nu} x_h \pmod{p},$$

$$\text{i.e., } \left(\sum_{h=1}^{h=\nu} x_h\right)^{p^r} \equiv \sum_{h=1}^{h=\nu} x_h \text{ (modd. } p, x_1^{p^r} - x_1, x_2^{p^r} - x_2, \ldots, x_\nu^{p^r} - x_\nu).$$

A special case of this theorem is when all the x's are made equal to unity. We then have

$$m^p \equiv m \pmod{p}.$$

18. *Reduction of an integral function into its irreducible factors.*

As in the case of common numbers the determination whether or not they may be resolved into divisors is completely accomplished by a finite number of operations, so by a finite number of operations we must show whether an integral function of x with integral coefficients has, or has not, rational divisors, *i.e.* whether a given quantity of the *integral realm* may or may not be represented as the product of two quantities belonging to this same realm.

Let $f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n \equiv 0 \pmod{p}$,

where the c's are integers and $\leqq p - 1$.

Also suppose that $(c_0, c_1, c_2, \ldots, c_n) \backsim m$,

so that $f(x) = m(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)$;

resolving m into its prime factors, we have

$$f(x) = p_1^{h_1} p_2^{h_2} \ldots (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n).$$

Now write $f(x)$ in the form

$$f(x) = \phi(x) \frac{f(x)}{\phi(x)} = \phi(x)\, \psi(x),$$

where $\phi(x)$ is of degree $\leqq \frac{1}{2}n$ or $\frac{1}{2}(n-1) = \nu$, according as n is even or odd.

We have therefore to consider as divisors of $f(x)$ only those functions whose degree are equal to or less than ν.

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If there does exist any divisor $\phi(x)$, then we may write the product $\phi(x)\psi(x)$ in the form

$$\phi(x)\psi(x) = x_1 - x_2 x^{\beta_1} - \ldots - x_m x^{\beta_m - \alpha_m} - \beta_1 x^{\alpha_1 - \beta_1} - \ldots - \beta_{m-1} x^{\alpha_{m-1} - \beta_{m-1}}),$$

where $m \leq n$, and where the α's and β's are integers, such that

$$\alpha_1 \beta_1 = \lambda_1, \quad \alpha_2 \beta_2 = \lambda_2, \quad \ldots \quad \alpha_m \beta_m = \lambda_m.$$

Hence the α's must be divisors of the λ's.

Let $r_1, r_2, r_3, \ldots, r_{n+1}$ be positive or negative integers differing from each other in the modulus p^ν, and form the $n+1$ equations

$$\begin{aligned} f(r_1) &= \phi(r_1)\psi(r_1) + p^\nu, \\ f(r_2) &= \phi(r_2)\psi(r_2) + p^\nu, \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \\ f(r_{n+1}) &= \phi(r_{n+1})\psi(r_{n+1}) + p^\nu. \end{aligned}$$

Corresponding to $r_1, r_2, \ldots, r_{n+1}$ we have a certain set of divisors for $f(x)$, which may be written

$$\begin{array}{llllll} f(r_1) & \lambda_1, & \lambda_1', & \lambda_1'', & \ldots, & \lambda_1^{(\mu)}, \\ f(r_2) & \lambda_2, & \lambda_2', & \lambda_2'', & \ldots, & \lambda_2^{(\mu)}, \\ f(r_3) & \lambda_3, & \lambda_3', & \lambda_3'', & \ldots, & \lambda_3^{(\mu)}, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f(r_{n+1}) & \lambda_{n+1}, & \lambda_{n+1}', & \lambda_{n+1}'', & \ldots, & \lambda_{n+1}^{(\mu)}. \end{array}$$

If, then, there is a divisor $\phi(r_\kappa)$, it must be one of the divisors that stands in the $\kappa+1^{\text{th}}$ line, and we have only a finite number of systems of divisors to discuss in order to find $\nu+1$ values for ϕr. If there are not $\nu+1$ values, then the given function is irreducible; if we can find $\nu+1$ values, then $\phi(x)$ is completely determined by Lagrange's formula of interpolation, and may be written in the form

$$\phi(x) = \sum_\kappa \phi(r_\kappa) \prod_{\substack{\kappa \\ h \gtrless \kappa}} \frac{x - r_h}{r_\kappa - r_h}.$$

$$\left(\begin{matrix} \kappa = 0, 1, 2, \ldots, \nu \\ h \gtrless \kappa \end{matrix}\right).$$

By continuing this process, it is seen that

$$f(x) = p_1^{k_1} \cdot p_2^{k_2} \ldots \phi_1(x)^{\kappa_1} \cdot \phi_2(x)^{\kappa_2} \ldots,$$

where the ϕ's are irreducible functions of x.

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19. Proceeding in the same manner with functions having two or more variables, we may write

$$f(x, y) = f_0(x) + yf_1(x) + y^2f_2(x) + \ldots + y^nf_n(x).$$

Considering this as a function of y alone, the functions of x taking the place of constants, we may proceed precisely as above in the case of one variable, and by induction applying this method to 3, 4, ..., m variables, we thus obtain the complete reduction of an integral function of m variables.*

20. It remains yet to show that this decomposition into factors is an unique one. To do this we have to prove the following lemma:

Lemma. If $\phi(x).\psi(x) \equiv 0$ (modd. $p, f(x)$),

and if $\phi(x)$ is not $\equiv 0$ (modd. $p, f(x)$),

then $\psi(x) \equiv 0$ (modd. $p, f(x)$),

where $f(x)$ is an irreducible function of x (mod. p).

1°. We consider the case where the modulus consists only of a prime number, and let

$$\phi(x) = a\,x^m + a_{m-1}x^{m-1} + \ldots,$$

$$\psi(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots.$$

Then $\phi(x) \equiv a_h x^m + a_{h-1}x^{m-1} + \ldots$ (mod. p),

$$\phi(x) \equiv b_k x^n + b_{k-1}x^{n-1} + \ldots \text{ (mod. } p),$$

where the coefficients have been reduced, so that they are at most equal to $p-1$.

Hence $\phi(x)\,\psi(x) \equiv a_h b_k x^{m+n} + \ldots \equiv 0$ (mod. p),

and since this congruence holds, whatever the value of x is, it follows that if

$$a_h \text{ is not} \equiv 0 \text{ (mod. } p),$$

then $$b_k \equiv 0 \text{ (mod. } p).$$

Therefore all the coefficients of x in the function $\psi(x)$ are congruent to zero, modulo p, and consequently

$$\psi(x) \equiv 0 \text{ (mod. } p).$$

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* Cf. again Kronecker (*Crelle*, Vol. 92, p. 11).

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was true in the case of one variable (Art. 16), it follows in the case of two variables, since Φ_k is not $\equiv 0 \pmod{F}$, that $\Psi_\kappa \equiv \Psi_{\kappa+1} \equiv \Psi_{\kappa+2} \equiv \ldots \equiv 0 \pmod{F}$, and similarly for any number of variables; and finally that

$$\psi = \psi_0 + \psi_1 x + \ldots \equiv 0 \pmod{F}.$$

21. Having proven the above lemma, it is easily shown that the decomposition of $f(x)$ (Art. 18) is an unique one.

For suppose that another decomposition were possible, and let it be written

$$f(x) = g_1^{h_1'} . g_2^{h_2'} \ldots f(x)^{\kappa_1'} . f_2(x)^{\kappa_2'} \ldots .$$

Then

$$p_1^{h_1} . p_2^{h_2} \ldots \phi_1(x)^{\kappa_1} . \phi_2(x)^{\kappa_2} \ldots \equiv 0 \pmod{g_1^{h_1'} . g_2^{h_2'} \ldots f_1(x)^{\kappa_1'} . f_2(x)^{\kappa_2'} \ldots};$$

and therefore certainly $\equiv 0 \pmod{g_1^{h_1'}}$.

From the above lemma this $g_1^{h_1'}$ is contained in some one of the p's. But since they are all prime numbers, it follows that it must be equal to some one of the p's.

Similarly for the functions of x.

This can be extended to functions of 2, 3, ... n variables without any difficulty; so that the decomposition of integral functions of any number of variables into their irreducible factors is an unique one.

22. *The Greatest Common Divisor of integral functions.*

It is clear that, after having reduced two functions to their irreducible factors, the G. C. D. of these two functions is the product of the irreducible factors common to both functions.

Another method of finding the G. C. D. is as follows:

Let the two functions be

$$f_1(x) = a_0 + a_1 x + \ldots + a_m x^m,$$

and

$$f_2(x) = b_0 + b_1 x + \ldots + b_n x^n,$$

where the a's and b's are integers, and where $m \gtrless n$.

Divide $f_1(x)$ by $f_2(x)$, and let the quotient be $g(x)$ and the remainder $r(x)$, so that

$$f_1(x) = g(x) f_2(x) + r(x).$$

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The *first* set of relations shows that any of the functions of x multiplied perhaps by a constant factor is divisible by a modular system, consisting of any two successive and later following functions of x; the *second* set shows that any one of the functions of x is divisible by a modular system consisting of any two successive and preceding functions of x.

Going backward in the first set of relations, we find that

$$\begin{aligned} n_1 . n_2 . n_3 \dots n_{\nu-1} f_1(x) &\equiv 0 \pmod{f_\nu(x)}, \\ n_2 . n_3 \dots n_{\nu-1} f_2(x) &\equiv 0 \pmod{f_\nu(x)}, \\ \vdots \qquad\qquad & \vdots \\ n_{\nu-1} f_{\nu-1}(x) &\equiv 0 \pmod{f_\nu(x)}. \end{aligned}$$

Therefore

$$\begin{aligned} s_1 f_1(x) &\equiv 0 \pmod{f_\nu(x)}, \\ s_2 f_2(x) &\equiv 0 \pmod{f_\nu(x)}, \end{aligned}$$

where s_1 and s_2 are integers.

Hence $$(s_1 f_1(x), s_2 f_2(x)) \equiv 0 \pmod{f_\nu(x)},$$

and when $s_1 = s_2 = 1$, we have the equivalence

$$(f_1(x), f_2(x)) \sim (f_\nu(x)).$$

As a rule, only in those cases where the coefficients are not considered can *two* modular systems be considered equivalent to *one*; and this is the reason why modular systems were not discovered sooner.

This result may be compared with the similar instance of a system of two equations with one unknown quantity, which represents nothing more than one equation.

24. If a function of n variables $G(x_1, x_2 \dots, x_n)$ can be represented as a linear homogeneous function of several integral functions $F_1(x_1, x_2, \dots, x_n)$, $F_2(x_1, x_2, \dots, x_n)$, $\dots$, $F_n(x_1, x_2, \dots, x_n)$, the coefficients of this function being also integral functions of $x_1, x_2, \dots, x_n$, *i.e.*, if

$$G(x_1, x_2, \dots, x_n) \equiv 0 \pmod{F_1, F_2, \dots, F_n},$$

then we call the system of elements $F_1, F_2, \dots, F_n$ with regard to the linear homogeneous functions that are formed from them a *modular system of the n^{th} kind.**

* Cf. Kronecker, *Crelle* Vol. XCII., p. 72. I translate the word "Stufe" employed by Prof. Kronecker by the English word "kind," in the absence of what seems to me a better word.

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Owing to the congruence

$$G(x_1, x_2, \dots x_n) \equiv 0 \,(\text{modd. } F_1, F_2, \dots F_n),$$

any one of the variables may be expressed as a function of the other $n-1$ variables.

Hence if R^h is the result of substituting x_h determined from any of the functions $F_1, F_2, \dots, F_n$, in any of the remaining $n-1$ functions of F, we have $n-1$ quantities $R', R'', \dots, R^{(n-1)}$. These R's contain $n-1$ independent variables, and form a modular system of the n^{th} kind.

A modular system of the 1st kind contains *therefore* no variable, and consists of a pure number; a modular system of the 2nd kind contains one variable, &c.

25. *The Reduction of Modular Systems of the second kind.*

When every element of a modular system has as a common factor an integer or an integral function of the variables that constitute the system, the system is said to be *mixed;* when there is no such common factor, we have a pure modular system.

As we may always take out the common factor, we will hereafter consider only pure modular systems.

26. Let us consider the modular system

$$(F_1(x), F_2(x), \dots, F_\mu(x))$$

in which the F's are integral functions of x.

Suppose that $f_1(x)$ is the G.C.D. of $F_1(x)$ and $F_2(x)$; then (Art. 5) we can add $f_1(x)$ to the modular system, thus having

$$(F_1(x), F_2(x), \dots, F_\mu(x), f_1(x)).$$

Again, suppose that $f_2(x)$ is the G.C.D. of $f_1(x)$ and $F_3(x)$, and form the new equivalent modular system with the addition of $f_2(x)$, &c. We must finally come to a numerical factor, which call m. Our original modular system thus takes the form

$$(F_1(x), F_2(x), \dots, F_\mu(x), f_1(x), f_2(x), \dots, m),$$

and this (see Art. 5) is equivalent to $(f_1(x), f_2(x), \dots, m)$.

The coefficients in $f_1(x), f_2(x)$ may be reduced so as to be at most equal to $m-1$.

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27. As an example let us consider the *reduction* of the equation

$$x^3 - 1 = 0.$$

This may be written

$$(x^2 + x + 1)(x - 1) = 0.$$

Forming the modular system

$$(x^2 + x + 1,\ 2x + 1),$$

and dividing $x^2 + x + 1$ by $2x + 1$, we have the quotient $\frac{1}{2}x$ and remainder $\frac{1}{2}x + 1$, so that

$$x^2 + x + 1 - \tfrac{1}{2}x(2x + 1) - \tfrac{1}{2}(x + 2) = 0,$$

or

$$2(x^2 + x + 1) - x(2x + 1) - (x + 2) = 0.$$

Hence $(x^2 + x + 1, 2x + 1) \sim (x^2 + x + 1, 2x + 1, x + 2)$.

And again

$$2x + 1 - 2(x + 2) + 3 = 0,$$

therefore

$$(x^2 + x + 1, 2x + 1, x + 2) \sim (x^2 + x + 1, 2x + 1, x + 2, 3).$$

But

$$2x + 1 - (x + 2) = x - 1 \text{ and } x(x - 1) + 2x + 1 = x^2 + x + 1.$$

so that

$$(x^2 + x + 1,\ 2x + 1) \sim (x - 1, 3),$$

therefore

$$3 \equiv 0 \text{ (modd. } x^2 + x + 1,\ 2x + 1).$$

Let us give x the definite value ρ, where $\rho = e^{\frac{2}{3}\pi i}$.

Then, since

$$\rho^2 + \rho + 1 = 0,$$

we have from the above congruence the following congruence:

$$3 \equiv 0 \text{ (mod. } 2\rho + 1).$$

Or, in other words, 3 is divisible by $2\rho + 1$,

or

$$3 = (2\rho + 1)\, g(\rho).$$

$g(\rho)$ may be determined as follows:

$$3 = 4\rho^2 - 1 = (2\rho + 1)(2\rho^2 + 1).$$

Hence

$$g(\rho) = 2\rho^2 + 1,$$

but

$$2\rho^2 + 1 = \rho(\rho - 1),$$

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Therefore $3 \equiv 0 (\text{mod. } \rho - 1)$,

$\rho - 1$ being also a divisor of 3.

Such factors are closely allied to the *ideal prime factors* which are first discussed by Kummer (*Crelle*, Vol. XXXV); also see Kronecker (*Abhand. der Königl. Akademie der Wissenschaften zu Berlin* 1859), Dedekind (*Vorlesungen über Zahlentheorie von Dirichlet* § 167), &c.

It is thus seen that the abstract properties of *ideal prime numbers* take a concrete algebraic form and are capable of being treated by the use of modular systems in a simple manner.

28. In the modular system

$$(m, f_1(x), f_2(x), \ldots),$$

suppose that one of the functions of $x, f_h(x)$ say, has the form

$$f_h(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots,$$

where a_n, the coefficient of the highest power of x, has no factor in common with m; then we can always multiply $f_h(x)$ by some constant factor, so that

$$\kappa . a_n \equiv 1 \text{ (mod. } m),$$

where κ is the constant factor.

If, however, the coefficient of the highest power of x in each of the functions of x, which constitute the modular system, has a common factor with m, we can yet form a function of x out of the functions $f_1(x), f_2(x)$, etc., in such a way that the coefficient of the highest power of x in this function shall be congruent to unity, modulo m.

29. It has been shown (Art. 5) that any function of x, $F(x)$, which is a linear function of the elements of a modular system may be added to this modular system without altering the equivalence.

Suppose that

$$F(x) = f_1(x) + x^{1+n_1} f_2(x) + x^{2+n_1+n_2} f_3(x) + \ldots$$

where $n_1, n_2, \ldots$ are the respective degrees of $f_1(x), f_2(x), \ldots$. In the function so formed all the terms have different degrees in x.

In $F(x)$ collect all those terms that contain p_κ, where p_κ is

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one of the prime factors of m, m being equal to $p_1 . p_2 \ldots$, so that

$$F(x) = \Phi_\kappa(x) - p_\kappa \Psi_\kappa(x),$$

where $\Phi_\kappa(x)$ does not contain p_κ as a factor, otherwise $F(x)$ would be divisible by p_κ.

Now, since $F(x) \equiv 0$ (our modular system),

it follows that

$$\Phi_\kappa(x) \equiv p_\kappa \Psi_\kappa(x) \text{ (modular system)};$$

hence $$\Phi_\kappa(x)^{h_\kappa} \equiv p_\kappa^{h_\kappa} \Psi_\kappa(x)^{h_\kappa} \text{ (modular system)}.$$

In $\Phi_\kappa(x)^{h_\kappa}$ the coefficient of highest power of x does not contain p_κ as a factor. Call this coefficient C_κ.

Then $C_\kappa \dfrac{m}{p_\kappa^{h_\kappa}}$ is also relative prime to ϕ_κ, and we can multiply $\Phi_\kappa(x)^{h_\kappa}$ by some constant B_κ, so that

$$B_\kappa C_\kappa \frac{m}{p_\kappa^{h_\kappa}} \equiv 1 \text{ (mod. } p_\kappa^{h_\kappa}).$$

We may thus form a series of functions of x, where the coefficients of the highest powers of x in each are respectively

$$B_1 C_1 \frac{m}{p_1^{h_1}} \equiv 1 \text{ (mod. } p_1^{h_1}), \quad B_2 C_2 \frac{m}{p_2^{h_2}} \equiv 1 \text{ (mod. } p_2^{h_2}), \text{ etc.}$$

It is also to be noted that

$$B_1 C_1 \frac{m}{p_1^{h_1}} \equiv 0 \text{ (mod. } p_2^{h_2}), \quad B_2 C_2 \frac{m}{p_2^{h_2}} \equiv 0 \text{ (mod. } p_1^{h_1}), \text{ etc.,}$$

$$\equiv 0 \text{ (mod. } p_3^{h_3}), \qquad \equiv 0 \text{ (mod. } p_3^{h_3}),$$

etc. etc.

We may multiply the above functions of x by such powers of x as will make them all of the same degree, so that the coefficient of the highest power of x in the function formed by the addition of all these functions will be

$$\left[B_1 C_1 \frac{m}{p_1^{h_1}} + B_2 C_2 \frac{m}{p_2^{h_2}} + \ldots \right],$$

and it is clear that this coefficient is congruent to unity, modulo m.

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Remark. The residues of an equation of the $n-1$ degree are at most equal to m^n in number, as may be seen from the following diagram:

$$\begin{array}{cccccc} c_0 & + & c_1x & + & c_2x^2 & +\ldots+ c_{n-1}x^{n-1}, \\ 0 & , & 0 & , & 0 & , \quad 0 , \\ 1 & , & 1 & , & 1 & , \quad 1 , \\ 2 & , & 2 & , & 2 & , \quad 2 , \\ \vdots & & \vdots & & \vdots & \quad \vdots \\ m-1, & & m-1, & & m-1, & \quad m-1. \end{array}$$

It is clear that, if n be the degree of the function $f_1(x)$ above, the number of different terms in a complete system of residues belonging to any modular system cannot be greater than m^n (Art. 42). The determination of this number in some possible manner is a problem as yet unsolved. In the case of common numbers it is known that if m is any integer, then the number of integers less than and prime to m is the well-known quantity $\phi(m)$.

30. The question arises how is it possible to ascertain whether a given function $R(x)$ is prime to a modular system?

We must add $R(x)$ to the system (cf. Art. 13), and the system thus formed must be equivalent to unity, *i.e.*, if $(m, f_1(x), f_2(x), \ldots)$ is the given system, then

$$(R(x), m, f_1(x), f_2(x), \ldots) \sim 1.$$

Suppose that

$$(R_1, R_2, \ldots, R_\nu)$$

is the complete system of residues with respect to the modular system $(m, f_1(x), f_2(x), \ldots)$.

Take out any of these R's, say R_i, and we will see that R_1R_i is also prime to this modular system, or, in other words,

$$(R_1R_i, m, f_1(x), f_2(x), \ldots) \sim 1.$$

Now

$$(R_1R_i, m, f_1(x), f_2(x), \ldots) \sim (R_1, m, f_1(x), f_2(x), \ldots) \times (R_i, m, f_1(x), f_2(x), \ldots).$$

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This last system may be regarded as consisting of the four component parts,

$$R_i R_{i'},\ R_1 (m, f_1(x), f_2(x), \ldots),\ R_{i'} (m, f_1(x), f_2(x), \ldots),$$

$$(m, f_1(x), f_2(x), \ldots)\ (m, f_1(x), f_2(x), \ldots).$$

The first and third of these combine into

$$R_{i'} (R_1, m, f_1(x), f_2(x), \ldots) \backsim R_{i'}.$$

The second and fourth combine into

$$(R_1, m, f_1(x), f_2(x), \ldots)\ (m, f_1(x), f_2(x), \ldots) \backsim$$

$$(m, f_1(x), f_2(x), \ldots).$$

There remains therefore only

$$R_{i'} \text{ and } (m, f_1(x), f_2(x), \ldots),$$

and these when combined are equivalent to unity.

Hence $(R_i R_{i'}, m, f_1(x), f_2(x), \ldots) \backsim 1.$

It is next to be noted that none of the products

$$R_{i'} R_1,\ R_{i'} R_2,\ \ldots,\ R_{i'} R_\nu$$

are congruent to each other with respect to our modular system.

For suppose that

$$R_{i'} R_\lambda \equiv R_{i'} R_\kappa \text{ (modd. } m, f_1(x), f_2(x), \ldots),$$

then $$R_{i'} (R_\lambda - R_\kappa) \equiv 0 \text{ (modd. } m, f_1(x), f_2(x), \ldots),$$

and consequently

$$(R_\lambda - R_\kappa)\ (R_{i'}, m, f_1(x), f_2(x), \ldots) \equiv 0$$

$$\text{(modd. } m, f_1(x), f_2(x), \ldots).$$

Therefore, since $(R_{i'}, m, f_1(x), f_2(x), \ldots) \backsim 1,$

$$R_\lambda - R_\kappa \equiv 0 \text{ (modd. } m, f_1(x), f_2(x), \ldots).$$

And since R_λ and R_κ are both relative prime to the modular system, the congruence can only be satisfied when $R_\lambda = R_\kappa$.

From this it follows that the system $(R_1, R_2, \ldots, R_\nu)$ is equivalent to the system $(R_i R_1, R_i R_2, \ldots, R_i R_\nu)$.

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Hence, also

$$\prod_{\kappa=1}^{\kappa=\nu}(x-R_\iota R_\kappa)\equiv\prod_{\kappa=1}^{\kappa=\nu}(x-R_\kappa)\,(\text{modd. } m, f_1(x), f_2(x), \ldots),$$

therefore $R_\iota^{\nu}\prod_{\kappa=1}^{\kappa=\nu}R_\kappa^{\nu}\equiv\prod_{\kappa=1}^{\kappa=\nu}R_\kappa\,(\text{modd. } m, f_1(x), f_2(x), \ldots)$.

Consequently

$$(R_\iota^{\nu}-1)\prod_{\kappa=1}^{\kappa=\nu}R_\kappa\equiv 0\ (\text{modd. } m, f_1(x), f_2(x), \ldots).$$

Hence also

$$(R_\iota^{\nu}-1)\left(\prod_{\kappa=1}^{\kappa=\nu}R_\kappa, m, f_1(x), f_2(x)\right)\equiv 0$$

$$(\text{modd. } m, f_1(x), f_2(x), \ldots),$$

and finally, since

$$\left(\prod_{\kappa=1}^{\kappa=\nu}R_\kappa, m, f_1(x), f_2(x), \ldots\right)\sim 1,$$

$$R_\iota^{\nu}-1\equiv 0\ (\text{modd. } m, f_1(x), f_2(x), \ldots),$$

which is a still *more general form of Fermat's theorem.*

The above applies literally to any modular system, provided that the system is *one of the highest kind*, that is, when in its most reduced form it contains one element more than the number of variables.

32. *Example in illustration of the residues of modular systems of the second kind.*

In the modular system $(2, x^2)$, $m^n=2^2=4$ and the four residues for the expression c_0+c_1x are

$$0+0.x=0,$$

$$0+1.x=\quad x,$$

$$1+1.x=1+x,$$

$$1+0.x=1.$$

To ascertain which of these residues are prime to the system we proceed as follows:

$$(1+x, 2, x^2)\sim(1+x, 2, -x)\sim(1+x, 2, 1)\sim 1.$$

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Hence 1 and $1+x$ are the residues prime to the system.

It is also seen that $(2, x)$ is a divisor of the system, the *complementary* divisor being $(2, x)$.

Remark. We have seen (Art. 20) that if the product of two modular systems is divisible by a prime modular system, then one of the modular systems in the product is divisible by the prime modular system; if, however, the modular system is not a prime system, then it is possible that two systems (M) and (N) when multiplied together form a system which is congruent to zero (mod. this system), without either (M) or (N) being congruent to zero (mod. this system) as is seen in the case of the example:

$$(2, x)(2, x) \equiv 0 \text{ (modd. } 2, x^2).$$

33. We have shown that any pure modular system of the second kind may be reduced to the form

$$(m, f_1(x), f_2(x), \ldots).$$

It is possible that more than one constant integer may appear in the system, but it was seen (Art. 6) that these may be reduced to one; also $f_1(x), f_2(x)$ may have a constant factor, so that we write the reduced modular system in its most general form

$$(m, m_1 f_1(x), m_2 f_2(x), \ldots),$$

where $m_1, m_2, \ldots$, are less than m.

If the integers $m_1, m_2, \ldots$, are not all divisors of m, we can replace the system $(m, m_1 f_1(x), m_2 f_2(x) \ldots)$ by an equivalent system in which all the constant factors of the functions are divisors of m. For we may add to the given system any element $n f_1(x)$, and we may so choose constants α and β that $\alpha m_1 + \beta n = t_1$, where t_1 is a divisor of m, and similarly for m_2, &c.

34. It is always possible to add to this system a function of x whose numerical coefficient is a divisor of m.

For, making the functions of x that constitute the system all of the same degree by multiplying by powers of x, we can so choose $\alpha_1, \alpha_2, \ldots$, that

$$\alpha_1 t_1 + \alpha_2 t_2 + \ldots \equiv 1 \text{ (mod. } m),$$

since $t_1, t_2, \ldots$ cannot all have a common divisor other than unity, otherwise the system would be a mixed system, and we would then consider only the pure system contained in this system.

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Call this function $f(x)$, and let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots ;$$

then if a_n is not a divisor of m, we can transform it into another function in which the coefficient of the highest power of x is a divisor of m.

Form another function similarly to $f(x)$, some of the elements $f_1(x), f_2(x)$, being multiplied throughout by some constant factor, so that the coefficient of the highest power of x in this new function, b_n, shall have no factor in common with a_n except unity.

Then we can always find constants α and β, such that

$$\alpha a_n + \beta b_n = d,$$

where d is a divisor of m.

We have thus shown in modular systems of the second kind that, *first*, one element can always be made an integer; *second*, that all the numerical factors of the elements (functions of x) that constitute the system are divisors of this integer; *third*, that a function of x may be added as an element to this system, the numerical factor of this function being unity; *fourth*, that the coefficient of the highest power of x in this last function is a divisor of m, the constant integer of the system.

It was also seen (Art. 28) that a function of x could be added to the system in which the coefficient of the highest power of x was unity.

35. If $m = \mu.\nu$, where μ and ν are relatively prime to each other, then

$$(m, m_1 f_1(x), m_2 f_2(x) \dots) \sim (\mu, m_1 f_1(x), m_2 f_2(x), \dots) \times (\nu, m_1 f_1(x), m_2 f_2(x), \dots).$$

For the right-hand side of this equivalence is composed of the four elements

$$\mu\nu ;\ (\mu m_1 f_1(x), \mu m_2 f_2(x), \dots) ;\ (\nu m_1 f_1(x), \nu m_2 f_2(x), \dots) ;$$
$$(m_1 f_1(x), m_2 f_2(x), \dots)\ (m_1, f_1(x), f_2(x), \dots).$$

We can add to the system which contains these four elements:

$$(\alpha\mu + \beta\nu)(m, m_1 f_1(x), m_2 f_2(x), \dots) = H \text{ (say)},$$

where α, β are so chosen that $\alpha\mu + \beta\nu = 1$.

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Now the second, third and fourth elements of our system contain H, and we may therefore strike them out, leaving

$$(\mu\nu, (\alpha\mu + \beta\nu)(m, m_1 f_1(x), m_2 f_2(x), \ldots)$$

which is equivalent to

$$(m, m_1 f_1(x), m_2 f_2(x), \ldots).$$

Therefore since $m = p_1^{h_1} . p_2^{h_2} \ldots$, any modular system

$$(m, m_1 f_1(x), m_2 f_2(x), \ldots)$$

can be decomposed into

$$(p_1^{h_1}, m_1 f_1(x), m_2 f_2(x))(p_2^{h_2}, m_1 f_1(x), m_2 f_2(x), \ldots).$$

So that we have only to consider the case where the constant of the modular system is a power of a prime

$$(p_\kappa^{h_\kappa}, m_1 f_1(x), m_2 f_2(x), \ldots).$$

And since $m_1, m_2, \ldots$, are divisors of $p_\kappa^{h_\kappa}$ (Art. 33), the most general form of modular systems that we have to consider is

$$(p^h, p^{h-1} g_1(x), p^{h-1} g_2(x), \ldots, p^{h-s} \phi_1(x), p^{h-s} \phi_2(x), \ldots,$$
$$p\chi_1(x), p\chi_2(x), \ldots, \psi_1(x), \psi_2(x), \ldots).$$

36. Before the consideration of the general case, we must consider in succession the cases where the prime number occurs to the first, second, third, ... powers, and then, by the method of induction, show that our reductions may be applied to the case where the prime number occurs to any power in the modular system.

When the prime enters of the first degree, the most general form of the system is

$$(p, f_1(x), f_2(x), \ldots, f_n(x)),$$

in which the functions of x are all integral, and arranged so that the degree of $f_1(x)$ is greater than that of $f_2(x)$, the degree of $f_2(x)$ being greater than that of $f_3(x)$, etc.

We can then take out of $f_1(x)$ all multiples of $f_2(x)$ as follows

$$\frac{f_1(x)}{f_2(x)} = q_1 - \frac{f_\lambda(x)}{f_2(x)},$$

q_1 being an integer or an integral function of x, and $f_\lambda(x)$ an integral function of x.

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Hence $$f_1(x) - q_1 f_2(x) + f_\lambda(x) = 0,$$

or $$f_\lambda(x) \equiv 0 \ (f_1(x), f_2(x));$$

and therefore $$(f_1(x), f_2(x)) \sim (f_2(x), f_\lambda(x)),$$

where $f_\lambda(x)$ is of lower degree than either $f_1(x)$ or $f_2(x)$.

Continuing this process, the system finally reduces to the form

$$(p,\ f_{\nu-1}(x),\ f_\nu(x)).$$

Here *either* $f_{\nu-1}(x)$ is divisible by $f_\nu(x)$, and the system is then equivalent to

$$(p,\ f_\nu(x)),$$

where $f_\nu(x)$ is an integral function of x, whose coefficients are all positive and less p; *or* f_ν is a pure number, and then the system is equivalent to unity, and ceases to be of interest.

37. We have shown that the modular system

$$(p,\ f_1(x),\ f_2(x),\ \ldots,\ f_n(x)),$$

can be reduced to the form

$$(p,\ F(x)).$$

It remains yet to see whether $(p, F(x))$ is the unique reduction of the system.

Suppose that another reduction $(p, F_1(x))$ were possible; then

$$(p, F_1(x)) \sim (p, F(x)),$$

and therefore $$F_1(x) \equiv \phi(x).F(x) \ (\text{mod. } p),$$

$$F(x) \equiv \psi(x).F_1(x) \ (\text{mod. } p),$$

where $\phi(x)$ and $\psi(x)$ are integral functions of x.

Multiplying both sides of the congruences together, we have

$$F_1(x).F(x) \equiv \phi(x)\,\psi(x)\,F_1(x)\,F(x) \ (\text{mod. } p).$$

Since neither $F(x)$ nor $F_1(x)$ is divisible by p, we may divide the above congruence by the product $F(x).F_1(x)$, so that

$$1 \equiv \phi(x)\,\psi(x) \ (\text{mod. } p).$$

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Suppose that the degrees of $\phi(x)$ and $\psi(x)$ are μ and ν respectively, so that the congruence takes the form

$$(\alpha_\mu x^\mu + \alpha_{\mu-1}x^{\mu-1} + \ldots)(\beta_\nu x^\nu + \beta_{\nu-1}x^{\nu-1} + \ldots) \equiv 1 \pmod{p}.$$

This congruence must exist when the variable vanishes, so that

$$\alpha_\mu\ \beta_\nu \equiv 0 \pmod{p}, \quad \alpha_{\mu-1}\beta_{\nu-1} \equiv 0 \pmod{p}, \text{ etc.},$$

$$\alpha_{\mu-1}\beta_\nu \equiv 0 \pmod{p}, \quad \alpha_{\mu-2}\beta_{\nu-2} \equiv 0 \pmod{p}, \text{ etc.},$$

$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Hence, the congruence is of the form

$$(0 + 0 + 0 + \ldots + 0 + 1)(0 + 0 + 0 + \ldots + 0 + 1) \equiv 1 \pmod{p},$$

or $\phi(x) = \psi(x) = 1$, and consequently $F_1(x) \equiv F(x) \pmod{p}$.

The coefficients of like powers of x in these two functions must be equal, since they are all less than p.

$F(x)$ is therefore equal to $F_1(x)$, and $(p, F(x))$ is the unique reduction of our modular system.

38. The modular system in which the prime occurs raised to the second power may be written in its most general form

$$(p^2, pf_1(x), pf_2(x), \ldots, pf_n(x), \phi_1(x), \phi_2(x), \ldots, \phi_m(x)) \ldots (1).$$

We have seen that

$$(p, \phi_1(x), \phi_2(x), \ldots, \phi_m(x)) \backsim (p, H(x)) \ldots\ldots (2),$$

in which $H(x)$ is the unique form which the function of x must take on the right-hand side of this equivalence.

From this equivalence it follows (see Art. 5)* that

$$H(x) = \sum_{\kappa=1}^{\kappa=m} \phi_\kappa(x)\, g_\kappa(x) + p\overline{H}(x) \ldots\ldots\ldots\ldots (a),$$

where $g_\kappa(x)$ and $\overline{H}(x)$ are integral functions of x.

Hence, we may add $H(x) - p\overline{H}(x)$ to the elements of (1), which then becomes

$$(p^2, pf_1(x), pf_2(x), \ldots, pf_n(x), \phi_1(x), \phi_2(x), \ldots,$$
$$\ldots, \phi_m(x), H(x) - p\overline{H}(x)),$$

* Any element which stands on one side of an equivalence is expressible as a linear homogeneous function of the elements that stand on the other side.

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[redacted]

[redacted] (3).

[redacted] (b),

[redacted]

[redacted] 3), we may [redacted] which now becomes

[redacted]

[redacted]

[redacted]

[redacted] that the functions of x can have [redacted]

[redacted]

[redacted]

[redacted] (4).

[redacted] similar result would have been obtained [redacted] the congruences, instead of the equations ϕ_i and f_i, the congruences

$$f_i(x_1, \ldots x_n) \equiv \sum_{k=1}^{\kappa} \phi_k(x_1, \ldots x_n) \pmod{p}; \quad \phi_k(x) \equiv H(x) h_k(x) \pmod{p}.$$

As the present paper is interested only in the reduction of systems to unique forms, I shall in the further discussion consider only congruences.

It is to be noted both here and in the further discussion of the systems that the powers of p serve as indices; as it is easy to see that no element of the system having p^h as constant factor can be associated in the reduction with another element having p^k as constant factor, where $h \leq \kappa$. Following the above reduction backward, it is seen that (4) is the [redacted] of the modular system (1).

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39. *On the reduction of modular systems where the prime enters to the third degree.* The system may be written

$$(p^3, p^2 f_1(x), p^2 f_2(x), \ldots, p\phi_1(x),$$
$$p\phi_2(x), \ldots, \psi_1(x), \psi_2(x), \ldots) \ldots\ldots\ldots\ldots (1).$$

We have seen that

$$(p^3, p\phi_1(x), p\phi_2(x), \ldots, \psi_1(x), \psi_2(x), \ldots) \sim$$
$$(p^3, pF(x), G(x))^* \ldots\ldots (2),$$

and also that the right-hand side of this last equivalence is the unique reduction of the left-hand side.

Since each element on the one side of this equivalence is expressible as a linear homogeneous function of the elements on the other side, it follows that

$$G(x) \equiv \sum_\kappa \Psi_\kappa(x)\, \psi_\kappa(x) + p \sum_h \Phi_h(x)\, \phi_h(x) \pmod{p^3},$$

$$pF(x) \equiv \sum_\kappa \Psi_\kappa'(x)\, \psi_\kappa(x) + p \sum_h \Phi_h'(x)\, \phi_h(x) \pmod{p^3},$$

where the indices κ and h extend over all the ψ's and ϕ's that constitute the system, and where $\Psi_\kappa(x)$, $\Phi_\kappa(x)$, $\Phi_h'(x)$, $\Psi_\kappa'(x)$ denote integral functions of x.

Also since

$$\psi_\kappa(x) \equiv G'(x)\, G(x) + pF'(x)\, F(x) \pmod{p^3},$$
$$p\phi_\kappa(x) \equiv G''(x)\, G(x) + pF''(x)\, F(x) \pmod{p^3},$$

in which $G'(x)$, $G''(x)$, $F'(x)$, $F''(x)$ are likewise integral functions of x, it is seen that we may add $G(x)$ and $pF(x)$ to the system (1), which becomes

$$(p^3, p^2 f_1(x), p^2 f_2(x), \ldots p\phi_1(x), p\phi_2(x), \ldots \psi_1(x), \psi_2(x), \ldots,$$
$$pF(x), G(x)),$$

and this is equivalent to

$$(p^3, p^2 f_1(x), p^2 f_2(x), \ldots pF(x), G(x)),$$

in which system we have struck out all the elements $p\phi_\kappa(x), \psi_h(x)$.

But

$$p(p^2, pf_1(x), pf_2(x), \ldots, F(x)) \sim p(p^2, pF_1(x), F(x)),$$

$(p^3, pF_1(x), F(x))$ being the unique reduction of

$$(p^3, pf_1(x), pf_2(x), \ldots, F(x)).$$

---

* Of course $F(x)$ is not the same as $F(x)$ of the preceding article.

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Hence (3) becomes

$$(p^s, p^s F_1(x), pF(x), G(x)) \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

It is seen that $G(x)$, $F(x)$, $F_1(x)$ correspond each respectively to an unique reduction. Hence (4) is the unique representation of (1).

40. When the prime occurs in the fourth degree, the modular system may be written

$$(p^4, p^3 f_1(x), p^3 f_2(x), \ldots, p^3\phi_1(x), p^3\phi_2(x), \ldots, p\psi_1(x), p\psi_2(x), \ldots,$$
$$\chi_1(x), \chi_2(x), \ldots) \ldots (1).$$

From the above

$$(p^3, p^3\phi_1(x), p^3\phi_2(x), \ldots, p\psi_1(x), p\psi_2(x), \ldots, \chi_1(x), \chi_2(x) \ldots)$$
$$\backsim (p^3, p^3 F(x), pG(x), H(x)),^*$$

where the right-hand side is the unique representation of the left-hand side. From this last equivalence we have

$$p^3 F(x) \equiv \Sigma k_\nu(x)\chi_\nu(x) + p\Sigma\Psi_\pi(x)\psi_\pi(x) + p^3\Sigma\Phi_\tau(x)\phi_\tau(x) (\text{mod. } p^3),$$

$$pG(x) \equiv \Sigma k_\nu'(x)\chi_\nu(x) + p\Sigma\Psi_\pi'(x)\psi_\pi(x) + p^3\Sigma\Phi_\tau'(x)\phi_\tau(x) (\text{mod. } p^3),$$

$$H(x) \equiv \Sigma k_\nu''(x)\chi_\nu(x) + p\Sigma\Psi_\pi''(x)\psi_\pi(x) + p^3\Sigma\Phi_\tau''(x)\phi_\tau(x) (\text{mod. } p^3);$$

and therefore

$$\chi_\nu(x) \equiv H'(x).H(x) + pG'(x).G(x) + p^3 F'(x)F(x) (\text{mod. } p^3),$$

$$p\psi_\pi(x) \equiv H''(x).H(x) + pG''(x).G(x) + p^3 F''(x)F(x) (\text{mod. } p^3),$$

$$p^3\phi_\tau(x) \equiv H'''(x).H(x) + pG'''(x).G(x) + p^3 F'''(x)F(x) (\text{mod. } p^3),$$

in which congruences $k_\nu(x)$, $k_\nu'(x)$, $\Psi_\pi(x)$, $H'(x)$, etc., are integral functions of x, and ν, π, τ are indices which belong to all the χ, ψ, and ϕ respectively.

From this set of congruences it is clear that $p^3 F(x), \phi G(x), H(x)$ may be added to (1), and that then all the terms $\chi_\nu(x)$,

---

* The different elements $\phi_1(x), \phi_2(x), \ldots, \psi_1(x), \ldots, \chi_1(x), \ldots, F(x), G(x)$, etc., have been used here for convenience, and have nothing to do with those used in the preceding article.

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$p\psi_\pi(x)$, $p^2\phi_\tau(x)$ may be omitted without altering the equivalence of the system, leaving

$$(p^n, p^2f_1(x), p^2f_2(x), \ldots, p^2F(x), pG(x), H(x)) \ldots\ldots\ldots (2).$$

But from the preceding article

$$p(p^n, p^2f_1(x), p^2f_2(x), \ldots, pF(x), G(x)) \backsim p(p^n, p^2F_1(x), pF(x), G(x)),$$

the right-hand side being the unique reduction of the left-hand side.

Hence (2) becomes

$$(p^n, p^2F_1(x), p^2F(x), pG(x), H(x)), \ldots,$$

and this is the unique reduction of the modular system (1).

41. The same demonstration may be applied when p occurs raised to the 5th power, etc.

Suppose that the proposition is true for the case when p is raised to the $n-1$ power. We will then have to show that the same is true when p occurs to the n^{th} power.

In order to condense the system as much as possible, in place of the elements $p^{n-h}\phi_1(x)$, $p^{n-h}\phi_2(x)$, $p^{n-h}\phi_3(x)$, etc., let us write $p^{n-h}\phi_\kappa(x)$, understanding that $p^{n-h}\phi_\kappa(x)$ is simply an abbreviation for all the terms $p^{n-h}\phi_1(x)$, $p^{n-h}\phi_2(x)$, ...

Suppose that

$$(p^{n-1}, p^{n-2}g_\kappa(x), p^{n-3}h_\lambda(x), \ldots, p^2r_\mu(x), ps_\nu(x), t_\pi(x))$$

$$\backsim (p^{n-1}, p^{n-2}G(x), p^{n-3}H(x), \ldots, p^2R(x), pS(x), T(x));$$

and also suppose that the right-hand member of this equivalence is the unique reduction of the left-hand member, then we must find the corresponding reduction for the modular system

$$(p^n, p^{n-1}f_\tau(x), p^{n-2}g_\kappa'(x), p^{n-3}h_\lambda'(x), \ldots, p^2r_\mu'(x), ps_\nu'(x), t_\pi'(x)) \ldots\ldots (1),$$

where $h_\lambda'(x)$ for example is simply an abbreviation for all the elements that have the common factor p^{n-3}.

By hypothesis

$$(p^{n-1}, p^{n-2}g_\kappa'(x), p^{n-3}h_\lambda'(x), \ldots, p^2r_\mu'(x)\, ps_\nu'(x), t_\pi'(x))$$

$$\backsim (p^{n-1}, p^{n-2}G_1(x), p^{n-3}H_1(x), \ldots, p^2R_1(x), pS_1(x), T_1(x)),$$

the right-hand member being the unique reduction of the left-hand member.

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Hence, it follows that

$$p^{n-2}G_1(x) \equiv \Sigma_\pi \tau_\pi(x)\, t_\pi'(x) + p\Sigma_\nu \sigma_\nu(x)\, s_\nu'(x)$$
$$+ p^2\Sigma_\mu \rho_\mu(x)\, r_\mu'(x) + \ldots + p^{n-2}\Sigma_\lambda \eta_\lambda(x)\, h_\lambda'(x) + p^{n-2}\Sigma_\kappa \gamma_\kappa(x)\, g_\kappa'(x),$$

$$p^{n-3}H_1(x) \equiv \Sigma_\pi \tau_\pi'(x)\, t_\pi'(x) + p\Sigma_\nu \sigma_\nu'(x)\, s_\nu'(x)$$
$$+ p^2\Sigma_\mu \rho_\mu'(x)\, r_\mu'(x) + \ldots + p^{n-3}\Sigma_\lambda \eta_\lambda'(x)\, h_\lambda'(x) + p^{n-2}\Sigma_\kappa \gamma_\kappa'(x)\, g_\kappa'(x),$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$pS_1(x) \equiv \Sigma_\pi \tau_\pi^{(n-3)}(x)\, t_\pi'(x) + p\Sigma_\nu \sigma_\nu^{(n-3)}(x)\, s_\nu'(x)$$
$$+ p^2\Sigma_\mu \rho_\mu^{(n-3)}(x)\, r_\mu'(x) + \ldots + p^{n-3}\Sigma_\lambda \eta_\lambda^{(n-3)}(x)\, h_\lambda'(x)$$
$$+ p^{n-2}\Sigma_\kappa \gamma_\kappa^{(n-3)}(x)\, g_\kappa'(x),$$

$$T_1(x) \equiv \Sigma_\pi \partial_\pi^{(n-2)}(x)\, t_\pi'(x) + p\Sigma_\nu \sigma_\nu^{(n-2)}(x)\, s_\nu'(x)$$
$$+ p^2\Sigma_\mu \rho_\mu^{(n-2)}(x)\, r_\mu'(x) + \ldots + p^{n-3}\Sigma_\lambda \eta_\lambda^{(n-2)}(x)\, h_\lambda'(x)$$
$$+ p^{n-2}\Sigma_\kappa \gamma_\kappa^{(n-2)}(x) g_\kappa'(x),$$
$$(\text{mod. } p^{n-1});$$

and also

$$t_\pi'(x) \equiv T_0(x)\, T_1(x) + pS_0(x)\, S_1(x)$$
$$+ p^2R_0(x)\, R_1(x) + \ldots + p^{n-3}H_0(x)\, H_1(x) + p^{n-2}G_0(x)\, G_1(x),$$

$$ps_\nu'(x) \equiv T_0'(x)\, T_1(x) + pS_0'(x)\, S_1(x)$$
$$+ p^2R_0'(x)\, R_1(x) + \ldots + p^{n-3}H_0'(x)\, H_1(x) + p^{n-2}G_0'(x)\, G_1(x),$$

$$\vdots$$

$$p^{n-3}h_\lambda'(x) \equiv T_0^{(n-3)}(x)\, T_1(x) + pS_0^{(n-3)}(x)\, S_1(x)$$
$$+ p^2R_0^{(n-3)}R_1(x) + \ldots + p^{n-3}H_0^{(n-3)}H_1(x) + p^{n-2}G_0^{(n-3)}(x)\, G_1(x),$$

$$p^{n-2}g_\kappa'(x) \equiv T_0^{(n-2)}(x)\, T_1(x) + pS_0^{(n-2)}(x)\, S_1(x)$$
$$+ p^2R_0^{(n-2)}R_1(x) + \ldots + p^{n-3}H_0^{(n-2)}(x)\, H_1(x) + p^{n-2}G_0^{(n-2)}(x)\, G_1(x)$$
$$(\text{mod. } p^{n-1}),$$

where the indices π, ν, μ, ..., λ, κ extend over all the quantities t', s', r', ..., h', g' respectively, and where

$$\tau, \sigma, \rho, \ldots, \eta, \gamma; \quad T_0, S_0, R_0, \ldots, H_0, G_0; \quad T_0', \text{ etc.}$$

denote integral functions of x.

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We may therefore add

$$p^{n-2}G_1(x),\ p^{n-3}H_1(x),\ \ldots,\ pS_1(x),\ T_1(x)$$

to the modular system (1) and then strike out all the terms $t_\pi'(x),\ ps_\pi'(x),\ \ldots,\ p^{n-3}h_\lambda'(x),\ p^{n-2}g_\kappa'(x)$, thus having

$$(p^n,\ p^{n-1}f_\tau(x),\ p^{n-2}G_1(x),\ p^{n-3}H_1(x),\ \ldots,\ pS_1(x),\ T_1(x))\ldots (2).$$

But by our hypothesis

$$p\,(p^{n-1},\ p^{n-2}f_\tau(x),\ p^{n-3}G_1(x),\ p^{n-4}H_1(x),\ \ldots,\ S_1(x))$$
$$\sim p\,(p^{n-1},\ p^{n-2}F_1(x),\ p^{n-3}G_1(x),\ p^{n-4}H_1(x),\ \ldots,\ S_1(x)).$$

The right-hand member is here also the unique reduction of the left-hand member.

Consequently (2) becomes

$$(p^n,\ p^{n-1}F_1(x),\ p^{n-2}G_1(x),\ p^{n-3}H_1(x),\ \ldots,\ pS_1(x),\ T_1(x)),$$

and this modular system as is seen from above is the unique reduction of the modular system (1).

42. It may happen in a modular system in which p^h is the constant term, that elements do not occur having as constant factors all powers of p less than h.

We may have for example the system

$$(p^2,\ f_1(x),\ f_2(x),\ \ldots).$$

From Art. 36,

$$(p, f_1(x), f_2(x), \ldots) \sim (p,\ F(x)),$$

and since any element standing on the one side of this equivalence is a linear homogeneous function of the elements standing on the other side, we have

$$f_\kappa(x) = F_\kappa'(x).F(x) + pT_\kappa(x),$$

where $F_\kappa'(x)$ and $T_\kappa(x)$ are integral functions of x.

Hence, instead of $f_\kappa(x)$ in our modular system we may substitute the elements $F_\kappa'(x)$, $F(x)$, $pT_\kappa(x)$, and the system can be then reduced after the manner given in Art. 38.

The same mode of procedure may be obviously applied whenever any set of elements is absent from a modular system.

Note. If $(p, f(x))$ is a modular system in which p is a prime number and $f(x)$ an irreducible function of the n^{th}

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degree in x, then the residues (incongruent quantities) of this modular system have the form

$$R(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1},$$

where the a's are all less than ϕ.

Hence the total number of these quantities is p^n. One of these quantities is divisible by the modular system, so that the whole number ν of quantities which are incongruent to each other, and are not divisible by the modular system, is $p^n - 1$.

Hence, from Art. 31.

$$(F(x))^{p^n-1} \equiv 1 \text{ (modd. } p, f(x)),$$

where $F(x)$ is any one of the residues.

That these $p^n - 1$ quantities are incongruent to each other is seen as follows.

Suppose that R_1 were congruent to R, then

$$R - R_1 = (a_0 - a_0') + (a_1 - a_1')\, x + (a_2 - a_2')\, x^2 + \ldots + (a_{n-1} - a'_{n-1})\, x^{n-1}.$$

If this expression is divisible by $(p, f(x))$, then

$$R - R_1 = Ap + Bf(x),$$

in which A and B are either integers or integral functions of x.

Since $R - R_1$ is of the $n-1$ degree, and $f(x)$ is of the n^{th} degree, the above relation cannot be true unless

$$Bf(x) = 0.$$

Therefore on this supposition

$$R - R_1 = Ap.$$

But $(a_0 - a_0')$, $(a_1 - a_1')$, ... are not divisible by p.

Hence the relation does not exist, and the two quantities are incongruent to each other.

4. *Lemma.* If $G(R) \equiv 0$ (modd. $p, f(x)$),

and if G is a function of the m^{th} degree in R, R being one of the residues above, and if $f(x)$ is an irreducible function of the n^{th} degree in x, then the congruence

$$G(R) \equiv 0 \text{ (modd. } p, f(x))$$

cannot have more than m roots.

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Suppose that Φ is one of the roots of this congruence, then

$$G(R) - G(\Phi) \equiv (R - \Phi)\, G_1(R) \equiv 0 \text{ (modd. } p, f(x)),$$

where $G_1(R)$ is at least one degree less in R than $G(R)$.

Suppose that the theorem is true for the $m-1$ degree, then it must also be true for the m^{th} degree.

For suppose that

$$(R - \Phi)\, G_1(R) \equiv 0 \text{ (modd. } p, f(x))$$

could have more than m roots, and that there exists another root Φ_0 so that

$$(\Phi_0 - \Phi)\, G_1(R_0) \equiv 0 \text{ (modd. } p, f(x)).$$

By our assumption, $G_1(R_0)$ cannot have more than $m-1$ values which satisfies the congruence

$$G_1(R_0) \equiv 0 \text{ (modd. } p, f(x)).$$

Therefore $$\Phi_0 - \Phi \equiv 0 \text{ (modd. } p, f(x)),$$

and hence Φ_0 is one of the m roots of the congruence

$$G(R) \equiv 0 \text{ (modd. } p, f(x)).$$

46. It is evident from the above lemma that

$$R^{p^n} - R \equiv 0 \text{ (modd. } p, f(x))$$

has p^n roots, of which one is zero.

Hence $$R^{p^n} - R \equiv R \prod_{\kappa} (R - R_\kappa) \text{ (modd. } p, f(x)),$$

where κ extends over the $p^n - 1$ residues of $F(x)$.

That is

$$(R^{p^n} - R) \equiv R\Pi (R - a_0 - a_1 x - a_2 x^2 - \ldots - a_{n-1} x^{n-1}) \text{ (modd. } p, f(x)),$$

in which $a_0, a_1, \ldots, a_{n-1}$ have all values 0, 1, 2, $p-1$, except the case where every a is equal to zero.

Equating coefficients of like powers of R on either side of the congruence, we have *the very general form of Wilson's theorem*

$$-1 \equiv \Pi (a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}) \text{ (modd. } p, f(x)),$$

where the a's have the same values as above.

Berlin, *May 25th*, 1894.

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NOTE ON HAMILTON'S NUMBERS.

By G. B. Mathews, M.A.

IN the paper on Hamilton's numbers (*Phil. Trans.* (*A*) 1887, p. 285) by Professor Sylvester and Mr. Hammond, the following table is exhibited:—

1	0	0	0	0	0	0	0 ...
	1	1	1	1	1	1	1 ...
		2	3	4	5	6	7 ...
			6	15	29	49	76 ...
				36	210	804	2449 ...
					876	24570	401134 ...
						408696	246382080 ...
							83762796636 ...,

the rule of formation being, that if

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,$$
$$b_1,\ b_2,\ b_3,\ \ldots$$

are any two consecutive lines in the table, then

$$b_s = \frac{a_1(a_1+1)(a_1+2)\ldots(a_1+s-1)(sa_1+1)}{(s+1)!} + \frac{a_1(a_1+1)(a_1+2)\ldots(a_1+s-2)}{(s-1)!}a_2 + \frac{a_1(a_1+1)(a_1+2)\ldots(a_1+s-3)}{(s-2)!}a_3 + \ldots + a_s.$$

The object of the present note is to call attention to the fact that the numbers in each row of the table form a recurring series, the general term of which may be calculated by a perfectly simple method, which, however, soon becomes

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laborious, and may perhaps be ultimately reduced to a better shape.

The fact above stated is an immediate consequence of the note in Lucas's *Théorie des Nombres*, I., p. 495. M. Lucas, with his usual sagacity, recognized in Prof. Sylvester's table a fragment of a more extensive table which proceeds according to a much simpler law. Lucas's table is

1	0	0	0	0	0	0 ...
1	1	1	1	1	1	1 ...
2	3	4	5	6	...	
1	5	9	14	20	...	
6	15	29	49	...		
5	21	50	99	...		
4	26	76	175	...		
3	30	106	281	...		
2	33	139	420	...		
1	35	174	594	...		
36	210	804	...			
35	246	1051	...			
34	281	1331	...			
⋮	⋮	⋮				

and the law of its formation is as follows.

The table is composed of successive stages, separated by the horizontal lines; in each stage every term is the sum of the number immediately above it, and all the numbers which precede the latter in its own row; thus take 175, the fourth number in the third row of the fourth stage, this is

$$99 + 50 + 21 + 5.$$

The same law holds good in passing from any stage to the next, except that the first column of any stage is composed of a set of numbers decreasing in their natural sequence down to 1. Thus, after completing the third stage, the head of the first column of the next stage is $1 + 5$, that is, 6; then we write down the terms 6, 5, 4, 3, 2, 1 for the first column, and then fill up the rest of the fourth stage according to rule.

It is proved without difficulty that the first line of the r^{th} stage in Lucas's table is the r^{th} line in Sylvester's table.

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Again, if

$$\ldots\ a_{s-1},\ a_s,\ a_{s+1}\ \ldots,$$

$$\ldots\ b_{s-1},\ b_s,\ b_{s+1}\ \ldots$$

are two successive rows of Lucas's table, then by the law of formation

$$\Delta b_s = a_{s+1},$$

and hence, if the first of these two rows is a recurring series, the second is also. It follows by induction that all the rows are recurring series.

It is convenient to modify Lucas's table a little, and write it as follows:

1	1	1	1	1	1 ...
1	2	3	4	5	6 ...
	2	5	9	14	20 ...
	1	6	15	29	49 ...
		6	21	50	99 ...
		5	26	76	175 ...
		4	30	106	281 ...
		3	33	139	420 ...
		2	35	174	594 ...
		1	36	210	804 ...
			36	246	1050 ...
			35	281	1331 ...
			⋮	⋮	⋮

If (r, n) denotes the n^{th} term of the r^{th} row of this table, it is found without difficulty that

$$(1, n) = 1,$$

$$(2, n) = n,$$

$$(3, n) = \tfrac{1}{2}n(n+3) = \frac{(n+1)(n+2)}{2!} - 1,$$

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$$(4, n) = \tfrac{1}{6}(n+2)(n^2+4n-3) = \frac{(n+1)(n+2)(n+3)}{3!} - (n+2),$$

$$(5, n) = \tfrac{1}{24} n (n+2)(n+5)(n+7)$$

$$= \frac{(n+2)(n+3)(n+4)(n+5)}{4!} - \frac{(n+3)(n+4)}{2!} + 1$$

$$= \frac{(n+2)(n+3)(n+4)(n+5)}{4!} - \frac{(n+2)(n+3)}{2!} - (n+2),$$

$$(6, n) = \tfrac{1}{720}(n+2)(n+3)(n^3+15n^2+54n-20)$$

$$= \frac{(n+2)(n+3)(n+4)(n+5)(n+6)}{5!} - \frac{(n+3)(n+4)(n+5)}{3!} + (n+3)$$

$$= \frac{(n+2)(n+3)(n+4)(n+5)(n+6)}{5!} - \frac{(n+2)(n+3)(n+4)}{3!} - \frac{(n+2)(n+3)}{2!},$$

$$(7, n) = \frac{(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)}{6!} - \frac{(n+2)(n+3)(n+4)(n+5)}{4!} - \frac{(n+2)(n+3)(n+4)}{3!} + 1$$

$$= (n+1)\left\{\frac{(n+2)(n+3)(n+4)(n+5)(n+12)}{6!} - \frac{n^2+8n+18}{3!}\right\}$$

$$= \tfrac{1}{5040}(n+1)(n^5+26n^4+239n^3+886n^2+1008n-720)$$

$$= (n+1)\left\{\frac{n(n+2)(n+7)(n+8)(n+9)}{6!} - 1\right\}.$$

Unfortunately I have not been able to discover the general law of these expressions; they have been given in various shapes, in the hope that some one else may discover a general formula.

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If $[r, n]$ is the n^{th} term of the r^{th} row of Sylvester's table,

$$[2, n] = 1,$$

$$[3, n] = n + 1,$$

$$[4, n] = \frac{(n+3)(n^2+6n+2)}{3!},$$

$$[5, n] = \frac{(n+2)(n+3)(n+4)(n+5)}{9!}$$

$$\{n^5 + 49n^4 + 905n^3 + 7511n^2 + 24798n + 3024\}$$

$$= \frac{(n+2)(n+3)(n+4)(n+5)}{9!}\Big\{(n+19)(n+9)(n+8)(n+7)(n+6)$$

$$- 7.8.9\,(n+7)(n+6) - 6.7.8.9\,(n+6) - 5.6.7.8.9\Big\};$$

but here again the general law is not obvious.

If h_r is the r^{th} Hamiltonian number, $[r, n]$ is a rational integral function of n of degree $(h_{r-1} - 2)$; thus, if $r = 5$, we have $h_{r-1} - 2 = 11 - 2 = 9$, and $[5, n]$ is, in fact, of the 9^{th} degree in n. Since $h_6 = 47$ it is clear that the calculation of $[6, n]$ by the method of differences would be very laborious; and all the series $[r, n]$ with $r > 6$ would be altogether beyond explicit calculation.

A RIGOROUSLY EUCLIDEAN DEMONSTRATION OF THE THEORY OF PARALLEL STRAIGHT LINES TO BE INTRODUCED IMMEDIATELY AFTER EUC. 1. 26.

By Thos. Cullovin.

LEMMA. *Let ABC, any rectilineal angle less than two right angles, be contained by the lines AB and BC, and bisected by the straight line BD, all three of unlimited length. It is to be shown that any straight line whatever drawn at right angles to BD will meet the containing lines AB and BC when produced.*

In AB take any point G (fig. 37). Make $BH = BG$. Join GH meeting BD in I. Then (I. 5 and 26) BIG

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and *BIH* will be right angles. In *BI* take any point *K* and draw through it a perpendicular to *BD*, this line must meet *AB* and *BC*, as otherwise it would meet *GH* forming triangles having each two right angles, which (I. 17) is impossible, and two straight lines would enclose a space. In the same way it can be shown that any other perpendicular to *BI* will meet *AB* and *BC*; and, generally, that if one perpendicular can be drawn to *BD* meeting also *AB* and *BC* (as *GH*), every perpendicular drawn at a point nearer to *B* than the former will meet the said lines. But an infinite number of perpendiculars may be drawn in the same way as *GH*. The enunciated property is therefore true for *part* of the line *BD*. That is to say, every perpendicular to an indefinite portion of the line *BD*, commencing at *B*, will meet *AB* and *BC*.

Again, conversely, if it be possible that any straight line *XZ* can be perpendicular to *BD* but never meet *AB* or *BC* however far produced, then *no* perpendicular drawn through a point more remote from *B* than the former (as *Q*) will meet the said lies, as otherwise it would, by its intersections with *XZ* produced, form triangles having each two right angles, and two straight lines would enclose a space.

The enunciated principle has been shown true for part of *BD*. But it is true for the whole of that line however far taken. For, if not, since every perpendicular to *BD* must either *meet* the lines *AB* and *BC* or *fail to meet* them when produced, *BD* taken to infinity, will consist of a finite part, *every* perpendicular to which will meet *AB* and *BC*, and an infinite part *no* perpendicular to which will meet the said lines however far produced. Let the finite part, if possible, end in the point *P* (fig. 38). That is to say, let the perpendicular to *BD* at *P* meet *AB*, *BC*, but *no* perpendicular to *BD* at any point more remote from *B* than *P* is, meet the said lines however far produced. Let *QR* be the perpendicular drawn through *P*, then (Euc. I. 26) $BQ=BR$. Take in *AB*, *S* a point more remote from *B* than *Q* is. Make $RT=QS$. Join *ST*, meeting *BD* in *V*. Then (Euc. I. 26 and 5) *BVS* and *BVT* will be right angles. Therefore through *V* a point more remote from *B* than *P* is, a perpendicular is drawn which meets *AB* and *BC*. But (hyp) no perpendicular passing through *BD* at a point more remote from *B* than *P* is, can meet the said lines. The contradiction is absurd. Therefore the line, all perpendiculars to which meet *AB* and *BC*, does not terminate at the point *P*. In the same way it can be shown that it does not terminate at any other point in *BD*.

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Therefore, it consists of the whole line BD however far taken. Wherefore, &c. Q.E.D.

Note. It is evident that no line of a *neutral* or *common* nature can be drawn through the supposed junction of the finite and infinite lines. Every perpendicular to BD must, when produced without limit, either *meet* AB and BC or *fail to meet* them. This kept in view, other forms of the demonstration will suggest themselves.

PROPOSITION. (*Euc. I. ax.* 12). *If a straight line meet*, &c.

(1) The case in which one of the angles is right follows directly from lemma.

(2) The case in which both are acute follows *a fortiori* from the foregoing.

(3) When one angle is obtuse. Let GBC (acute) + BCD (obtuse) or be less than 2 right. Make alternate angles $CBL = BCK$. Bisect BC in H (fig. 39). Let fall HL, HK, perpendiculars. These will (superposition) fall in the same line and the demonstration follows according to case I. Wherefore, &c. Q.E.D.

Professor Cayley remarks upon the above proof that there is an unwarranted assumption in the construction "take in AB, S a point more remote from B than Q is." For the critical point P may be such that there is not on AB any point S more remote from B than Q is, in other words the point P may be such that the point Q is at an infinite distance from P.

A proof is possible if we assume that BP being finite, BQ is also finite: but this is in effect Euclid's axiom in another form.

In answer, the author, introducing the symbol ϕ to mean "at a finite distance from B," proceeds as follows:

Let ABC be a rectilineal angle less than two right angles contained by the lines AB and BC of unlimited length and bisected by the straight line BD of unlimited length; any line drawn at right angles to the bisector ϕ, will meet the containing lines ϕ, when produced (fig. 40).

I show, as before, that an infinite number of perpendiculars to BD may be drawn ϕ by joining equal lengths in the

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containing lines. That all perpendiculars intermediate between those and B meet the containing lines ϕ. And that, consequently, every perpendicular to the bisector for a line of indefinite length meets the containing lines ϕ.

I next show, as before, that if any perpendicular fail to meet the containing lines ϕ, every perpendicular more remote from B must also fail.

Then, if the enunciated property be not true for any finite distance whatever in BD, that line taken to infinity must consist of a finite part *every* perpendicular to which meets the containing lines ϕ, and an infinite line *no* perpendicular to which will meet said lines ϕ. For a perpendicular may be drawn to any point, and each one must either *meet* the containing lines ϕ or *fail to meet* to meet them ϕ, no perpendicular of a *neutral* or *common* nature being possible. Assume P as the end of the finite line, and it may be shown, as before, that the perpendiculars do not cease to meet the containing lines ϕ at the point P.

EXAMPLES OF THE CHARACTERISTIC FUNCTION.

By R. A. Herman, M.A.

IN the Proceedings of the *London Mathematical Society*, Vol. VI., p. 183 (reprinted in *Collected Papers*, Vol. II., p. 381), Clerk Maxwell gave an investigation of Hamilton's Characteristic Function for a narrow beam of light. This was further developed by Mr. J. Larmor in two papers, also in the Proceedings of the *London Mathematical Society*, Vol. XX., p. 181, and Vol. XXIII., p. 165. In these papers the value of the characteristic function between two points is supposed known, and the properties of the pencil deduced in terms of the constants contained in that function. The aim of the following paper is to determine the values of these constants in certain cases.

Section 1. Coaxial Refracting Surfaces.

Let there be n surfaces, separating media of refractive indices $\mu_0, \mu_1, \ldots \mu_n$, and so placed that the normals at their vertices are all in the same straight line, which is the direction of the axis of a small incident pencil. Let the distances between the vertices be $\tau_1, \tau_2 \ldots \tau_n$, and let the equation of any surface referred to its vertex as origin and the axis of the pencil as axis of ζ be of the form

$$2\zeta = r\xi^2 + 2s\xi\eta + t\eta^2.$$

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Take the vertex of the first surface as origin of coordinates x, y, z, the axis of z being in the opposite direction to that of the incident pencil; and take the vertex of the last surface as origin of coordinates x', y', z', the axis of z' being in the same direction as that of the pencil. Let A be any point near the axis in the first medium, B in the last; $P_1, P_2, \ldots, P_n$ the points on the refracting surfaces where a ray passing from A to B is refracted.

Then the characteristic function

$$V = \mu_0 AP + \mu_1 P_1 P_2 + \ldots + \mu_n PB,$$

$$= \mu_0\{(\xi_1 - x)^2 + (\eta_1 - y)^2 + (\zeta_1 - z)^2\}^{\frac{1}{2}}$$
$$+ \mu_1\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\tau_1 + \zeta_1 - \zeta_2)^2\}^{\frac{1}{2}} + \ldots$$
$$+ \mu_n\{(\xi_n - x')^2 + (\eta_n - y')^2 + (\zeta_n + z')^2\}^{\frac{1}{2}}.$$

$$= \mu_0\left[z + \frac{x^2 + y^2}{2z} - \frac{x\xi_1 + y\eta_1}{z} + \frac{\xi_1^2 + \eta_1^2}{2z} - \tfrac{1}{2}(r_1\xi_1^2 + 2s_1\xi_1\eta_1 + t_1\eta_1^2)\right]$$

$$+ \mu_1\left[\tau_1 + \tfrac{1}{2}\left\{\frac{\xi_1^2 + \eta_1^2}{\tau_1} + r_1\xi_1^2 + 2s_1\xi_1\eta_1 + t_1\eta_1^2 + \frac{\xi_2^2 + \eta_2^2}{\tau_2}\right.\right.$$
$$\left.\left. - (r_2\xi_2^2 + 2s_2\xi_2\eta_2 + t_2\eta_2^2) - 2\frac{\xi_1\xi_2 + \eta_1\eta_2}{\tau_1}\right\}\right] + \ldots$$

$$+ \mu_n\left[z' + \frac{x'^2 + y'^2}{2z'} - \frac{x'\xi_n + y'\eta_n}{z'} + \frac{\xi_n^2 + \eta_n^2}{2z'}\right.$$
$$\left. + \tfrac{1}{2}(r_n\xi_n^2 + 2s_n\xi_n\eta_n + t_n\eta_n^2)\right],$$

where only squares and products of the small quantities ξ, η are retained.

Now A and B being given points, the points $P_1, P_2, \ldots P_n$ are given by the equation $\frac{\partial V}{\partial \xi} = 0$, $\frac{\partial V}{\partial \eta} = 0$ for all suffixes. These equations are linear and determine P_1, &c. uniquely. Let V_2 be the part of V which is of the second order in ξ and η; then

$$\frac{\partial V_2}{\partial \xi_1} = \mu_0\frac{x}{z}, \quad \frac{\partial V_2}{\partial \eta_1} = \mu_0\frac{y}{z}, \quad \frac{\partial V_2}{\partial \xi_n} = \mu_n\frac{x'}{z'}, \quad \frac{\partial V_2}{\partial \eta_n} = \mu_n\frac{y'}{z'},$$

and $\frac{\partial V_2}{\partial \xi} = \frac{\partial V_2}{\partial \eta} = 0$ for all other suffixes.

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Hence

$$V=\mu_0 s+\mu_1\tau_1+\ldots+\mu_n s'+\mu_0\left(\frac{x^2+y^2}{2s}-\frac{x\xi_1+y\eta_1}{2s}\right)$$

$$+\mu_n\left(\frac{x'^2+y'^2}{2s'}-\frac{x'\xi_n+y'\eta_n}{2s'}\right)+\tfrac{1}{2}\left(\xi_1\frac{\partial V_2}{\partial\xi_1}+\ldots+\eta_n\frac{\partial V_2}{\partial\eta_n}\right).$$

$$=\mu_0 s+\mu_1\tau_1+\ldots+\mu_n s'+\tfrac{1}{2}\left\{\mu_0\frac{x(x-\xi_1)+y(y-\eta_1)}{s}\right.$$

$$\left.+\mu_n\frac{x'(x'-\xi_n)+y'(y'-\eta_n)}{s'}\right\}.$$

Hence only $\xi_1, \eta_1, \xi_n, \eta_n$ need be determined. Let them be found as linear functions of x, y, x', y', and on substitution let

$$V=\mu_0 s+\mu_1\tau_1+\ldots+\mu_n s'+\tfrac{1}{2}\{c_{11}x^2+2c_{12}xy+c_{22}y^2$$

$$+c'_{11}x'^2+2c_{12}'x'y'+c_{22}'y'^2-2(d_{11}xx'+d_{22}yy')+2(d_{12}xy'+d_{21}x'y)\}.$$

The values of these coefficients are determined below; there is some advantage in having a minus sign before the two terms $d_{11}xx'+d_{22}yy'$.

If A be the origin of a full pencil, then the direction-cosines of the particular ray which passes through B, a point taken arbitrarily, are

$$-\frac{1}{\mu_0}\frac{\partial V}{\partial x},\ -\frac{1}{\mu_0}\frac{\partial V}{\partial y},\ -\frac{1}{\mu_0}\frac{\partial V}{\partial s}.$$

But if the pencil is determined by its orthotomic surface $s=\tau_0+\frac{1}{2}(ax^2+2hxy+by^2)$, there is only one ray through A, and its direction cosines are

$$ax+hy,\ hx+by,\ -1.$$

Hence the equations

$$-\mu_0(ax+hy)=c_{11}x+c_{12}y-d_{11}x'+d_{12}y',$$

$$-\mu_0(hx+by)=c_{12}x+c_{22}y+d_{21}x'-d_{22}y',$$

serve to determine the coordinates of the point B through which this ray passes. In the last medium let the orthotomic surface which passes through B be

$$s'=\tau_n-\tfrac{1}{2}(a'x'^2+2h'x'y'+b'y'^2);$$

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The equations for ξ, η, &c. are

$$\left[\mu_1\left(\frac{1}{\tau_1}+r_1\right)+\mu_0\left(\frac{1}{g}-r_1\right)\right]\xi_1+(\mu_1-\mu_0)\,s_1\eta_1-\frac{\mu_1}{\tau_1}\xi_2=\mu_0\frac{x}{g},$$

$$(\mu_1-\mu_0)\,s_1\xi_1+\left[\mu_1\left(\frac{1}{\tau_1}+t_1\right)+\mu_0\left(\frac{1}{z}-t_1\right)\right]\eta_1-\frac{\mu_1}{\tau_1}\eta_2=\mu_0\frac{y}{g},$$

$$-\frac{\mu_1}{\tau_1}\xi_1+\left[\mu_2\left(\frac{1}{\tau_2}+r_2\right)+\mu_1\left(\frac{1}{\tau_1}-r_2\right)\right]\xi_2$$

$$+(\mu_2-\mu_1)\,s_2\eta_2-\frac{\mu_2}{\tau_2}\xi_2=0,$$

.....,

and

$$-\frac{\mu_{n-1}}{\tau_{n-1}}\xi_{n-1}+\left[\mu_n\left(\frac{1}{g'}+r_n\right)+\mu_{n-1}\left(\frac{1}{\tau_{n-1}}-r_n\right)\right]\xi_n$$

$$+(\mu_n-\mu_{n-1})\,s_n\eta_n=\mu_n\frac{x'}{g'},$$

$$-\frac{\mu_{n-1}}{\tau_{n-1}}\eta_{n-1}+(\mu_n-\mu_{n-1})\,s_n\xi_n$$

$$+\left[\mu_n\left(\frac{1}{g'}+t_n\right)+\mu_{n-1}\left(\frac{1}{\tau_{n-1}}-t_n\right)\right]\eta_n=\mu_n\frac{y'}{g'}.$$

Put $a_{11}+\mu_0/z$, a_{12}, a_{13}, ..., for the coefficients of the terms in the first line; a_{21}, $a_{22}+\mu_0/z$, a_{23}, ..., for the coefficients of the second line, and so on. That is,

$$a_{11}=(\mu_1-\mu_0)\,r_1+\mu_1/\tau_1,\ a_{22}=(\mu_1-\mu_0)\,t_1+\mu_1/\tau_1,$$

$$a_{33}=(\mu_2-\mu_1)\,r_2+\mu_1/\tau_1+\mu_2/\tau_2,\ a_{44}=(\mu_2-\mu_1)\,t_2+\mu_1/\tau_1+\mu_2/\tau_2,\ \&c.,$$

$$a_{12}=(\mu_1-\mu_0)\,s_1,\ a_{13}=-\mu_1/\tau_1,\ a_{14}=a_{15}=\ldots a_{1.2n}=0,$$

$$a_{21}=(\mu_1-\mu_0)\,s_1,\ a_{24}=-\mu_1/\tau_1,\ a_{25}=a_{26}=\ldots a_{2.2n}=0,$$

$$a_{2n.1}=a_{2n.2}=\ldots a_{2n.2n-3}=0;\ a_{2n.2n-2}=-\mu_{n-1}/\tau_{n-1},$$

$$a_{2n.2n-1}=(\mu_n-\mu_{n-1})\,s_n,$$

and $$a_{2n.2n}=(\mu_n-\mu_{n-1})\,t_n+\mu_{n-1}/\tau_{n-1},$$

and let $D=\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, \ldots, & a_{1.2n} \\ a_{21}, & a_{22}, & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{2n.1}, & & & a_{2n.2n} \end{vmatrix}$.

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Then D is really a symmetrical determinant, in which the slants immediately adjacent to the leading diagonal are

$$(\mu_1 - \mu_0)s_{11}, 0, (\mu_2 - \mu_1)s_{22}, 0 \ldots, (\mu_n - \mu_{n-1})s_{nn};$$

and the slants next to those are

$$-\mu_1/\tau_{11}, -\mu_1/\tau_{12}, -\mu_2/\tau_{22}, -\mu_2/\tau_{23}, \ldots -\mu_{n-1}/\tau_{n-1,n}, -\mu_{n-1}/\tau_{n-1,n}.$$

The full notation is adopted for D to enable us to expand it more easily.

Now, on solving the equations above, we obtain in the common denominator

$$\Delta = f(a_{11} + \mu_0/z, a_{22} + \mu_0/z, a_{2n-1,2n-1} + \mu_n/z', a_{2n,2n} + \mu_n/z')$$

$$= D + \frac{\mu_0}{z}\left(\frac{\partial}{\partial a_{11}} + \frac{\partial}{\partial a_{22}}\right) D + \frac{\mu_n}{z'}\left(\frac{\partial}{\partial a_{2n-1,2n-1}} + \frac{\partial}{\partial a_{2n,2n}}\right) D$$

$$+ \ldots + \frac{\mu_0^2 \mu_n^2}{z^2 z'^2} \frac{\partial^4 D}{\partial a_{11}\partial a_{22}\partial a_{2n-1,2n-1}\partial a_{2n,2n}},$$

since every element occurs only linearly in D.

We later shall make z and z' zero, so that it is convenient to write

$$\Delta = \frac{1}{z^2 z'^2}\left[\mu_0^2 \mu_n^2 (a_{11}a_{22}a_{2n-1,2n-1}a_{2n,2n})\right.$$

$$\left. + \text{terms vanishing with } z, z'\right],$$

where $(a_{11}a_{22}a_{2n-1,2n-1}a_{2n,2n})$ is written for the coefficient in D of the product of these four elements.

Solving for ξ_1, we have ξ_1 proportional to

$$\left[\mu_0 \frac{x}{z} \frac{\partial}{\partial a_{11}} + \mu_0 \frac{y}{z} \frac{\partial}{\partial a_{21}} + \mu_n \frac{x'}{z'} \frac{\partial}{\partial a_{2n-1,1}} + \mu_n \frac{y'}{z'} \frac{\partial}{\partial a_{2n,1}}\right]\Delta,$$

and in V we have to calculate $\mu_0 \frac{x}{z}(x - \xi_1)$.

Now

$$(x - \xi_1)\Delta = \left[x . \exp\left(\frac{\mu_0}{z} \frac{\partial}{\partial a_{11}} + \frac{\mu_0}{z} \frac{\partial}{\partial a_{22}} + \frac{\mu_n}{z'} \frac{\partial}{\partial a_{2n-1,2n-1}}\right.\right.$$

$$\left.\left. + \frac{\mu_n}{z'} \frac{\partial}{\partial a_{2n,2n}}\right) D - \frac{\mu_0 x}{z} \frac{\partial}{\partial a_{11}} . \exp\left(\frac{\mu_0}{z} \frac{\partial}{\partial a_{22}} + \ldots\right) D\right]$$

$$- \text{the other terms in } y, x', y'.$$

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But since

$$\exp\left(\frac{\mu_0}{z}\frac{\partial}{\partial a_{11}}\right)D=\left(1+\frac{\mu_0}{z}\frac{\partial}{\partial a_{11}}\right)D,$$

this reduces to

$$\begin{aligned}(x-\xi_1)\,\Delta = x\exp&\left(\frac{\mu_0}{z}\frac{\partial}{\partial a_{22}}+\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n-1,2n-1}}+\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n,2n}}\right)D\\ &-\frac{\mu_0 y}{z}\frac{\partial}{\partial a_{21}}\exp\left(\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n-1,2n-1}}+\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n,2n}}\right)D\\ &-\frac{\mu_n x'}{z'}\frac{\partial}{\partial a_{2n-1,1}}\exp\left(\frac{\mu_0}{z}\frac{\partial}{\partial a_{22}}+\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n,2n}}\right)D\\ &-\frac{\mu_n y'}{z'}\frac{\partial}{\partial a_{2n,1}}\exp\left(\frac{\mu_0}{z}\frac{\partial}{\partial a_{22}}+\frac{\mu_n}{z'}\frac{\partial}{\partial a_{2n-1,2n-1}}\right)D\end{aligned}$$

$$\begin{aligned}=\frac{\mu_0\mu_n^2}{zz'^2}\,[&x\,(a_{22}a_{2n-1,2n-1}a_{2n,2n})-y\,(a_{21}a_{2n-1,2n-1}a_{2n,2n})\\ &-x'\,(a_{2n-1,1}a_{22}a_{2n,2n})-y'\,(a_{2n,1}a_{22}a_{2n-1,2n-1})]\\ &+(\text{terms vanishing with } z,\,z')/zz'^2.\end{aligned}$$

In the same way we find

$$\begin{aligned}(y-\eta_1)\,\Delta=\frac{\mu_0\mu_n^2}{zz'^2}\,[&-x\,(a_{1,2}a_{2n-1,2n-1}a_{2n,2n})+y\,(a_{11}a_{2n-1,2n-1}a_{2n,2n})\\ &-x'\,(a_{2n-1,2}a_{11}a_{2n,2n})-y'\,(a_{2n,2}a_{11}a_{2n-1,2n-1}]\\ &+(\text{terms vanishing with } z,\,z')/zz'^2.\end{aligned}$$

So also

$$\begin{aligned}(x'-\xi_n)\,\Delta=\frac{\mu_0^2\mu_n}{z^2z'}\,[&-x\,(a_{1,2n-1}a_{22}a_{2n,2n})-y\,(a_{2,2n-1}a_{11}a_{2n,2n}]\\ &+x'\,(a_{11}a_{22}a_{2n,2n})-y'\,(a_{2n,2n-1}a_{11}a_{22})]+\&c.,\end{aligned}$$

$$\begin{aligned}(y'-\eta_n)\,\Delta=\frac{\mu_0^2\mu_n}{z^2z'}\,[&-x\,(a_{1,2n}a_{22}a_{2n-1,2n-1})-y\,(a_{2,2n}a_{11}a_{2n-1,2n-1})\\ &-x'\,(a_{2n-1,2n}a_{11}a_{22})+y'\,(a_{11}a_{22}a_{2n-1,2n-1})]+\&c.\end{aligned}$$

Because D is a symmetrical determinant, several of these coefficients are equal in pairs, *e.g.*, the coefficient of y in $(x-\xi_1)\,\Delta$ and of x in $(y-\eta_1)\,\Delta$ are equal; and the reciprocal relation holds generally.

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Hence on substituting in the terms of the second degree in V, viz.,

$$\tfrac{1}{2}\left\{\mu_0 \frac{x(x-\xi_1)+y(y-\eta_1)}{z} + \mu_n \frac{x'(x'-\xi_n)+y'(y'-\eta_n)}{z'}\right\},$$

and making z, z' zero, which comes to drawing the two orthotomic surfaces with their vertices on the boundaries of the media in which they lie, we find that these terms are

$$\tfrac{1}{2}[c_{11}x^2 + 2c_{12}xy + c_{22}y^2 + c_{11}'x'^2 + 2c_{12}'x'y' + c_{22}'y'^2 - 2d_{11}xx' - 2d_{22}yy' + 2d_{12}xy' + 2d_{21}x'y].$$

These coefficients can be written down at once by the use of D. They all have as common denominator the determinant of $(2n-4)$ rows and columns obtained by cutting out the first and second, the last and last but one of both rows and columns. The numerators can be found thus: associate x with the first row and column of D, y with the second row and column, x' with those last but one, and y' with the last. For the coefficient c_{11} of x^2 cancel the y, x', y' rows and columns, and so for each of the squares.

For the coefficient c_{12} of xy cancel the x row and y column and the x' and y' rows and columns; and so for all the other products. All the coefficients are these determinants of $(2n-3)$ rows and columns with the sign plus, except those of xx' and yy' where the minus sign is introduced.

No loss of generality arises from drawing the orthotomic surfaces with their vertices at the boundaries of the media in which they lie: since if $z = \frac{1}{2}(ax^2 + 2hxy + by^2)$ be the orthotomic surface through the origin, the surface through $z = \tau_0$ is

$$z - \tau_0 = \tfrac{1}{2}\frac{ax^2 + 2hxy + by^2 - (ab-h^2)\tau_0(x^2+y^2)}{1-(a+b)\tau_0 + (ab-h^2)\tau_0^2}.$$

Example.

Let the principal planes of curvature of all the refracting surfaces be coincident; the principal planes of the incident light may or may not coincide with them. Taking the former as planes of reference, the terms of type $s\xi\eta$ will be absent from V, and the equations for ξ and η will be entirely independent of each other. Taking those for ξ, they are

$$\begin{aligned}
&\{(\mu_1-\mu_0)r_1 + \mu_1/\tau_1 + \mu_0/z\}\,\xi_1 - \xi_2\mu_1/\tau_1 &&= \mu_0 x/z\\
&\quad - \xi_1\mu_1/\tau_1 + \{(\mu_2-\mu_1)r_2 + \mu_2/\tau_2 + \mu_1/\tau_1\}\,\xi_2 &&= 0\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-\xi_{n-1}\mu_{n-1}/\tau_{n-1} + \{(\mu_n-\mu_{n-1})r_n + \mu_{n-1}/\tau_{n-1} + \mu_n/z'\}\,\xi_n = \mu_n x'/z'.
\end{aligned}$$

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Put $a_{11}, a_{12}, \ldots, a_{1n}$ for the coefficients of the terms in the first line, $a_{21}, a_{22}, \ldots, a_{2n}$ for those of the second line, and so on, and let D denote the determinant of n rows and columns,

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots & & \ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots, & & a_{nn} & & \end{vmatrix}.$$

Then, on solving these equations, we have

$$\xi_1 = \frac{\mu_0 \frac{x}{z} \frac{\partial}{\partial a_{11}} \left(1 + \frac{\mu_n}{z'} \frac{\partial}{\partial a_{nn}}\right) D + \mu_n \frac{x'}{z'} \frac{\partial D}{\partial a_{n1}}}{D + \frac{\mu_0}{z} \frac{\partial D}{\partial a_{11}} + \frac{\mu_n}{z'} \frac{\partial D}{\partial a_{nn}} + \frac{\mu_0 \mu_n}{zz'} \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}}},$$

and

$$\mu_0 \frac{x - \xi_1}{z} = \frac{\mu_0 \frac{x}{z} \left(D + \frac{\mu_n}{z'} \frac{\partial D}{\partial a_{nn}}\right) - \frac{\mu_0 \mu_n x'}{zz'} \frac{\partial D}{\partial a_{n1}}}{D + \frac{\mu_0}{z} \frac{\partial D}{\partial a_{11}} + \frac{\mu_n}{z'} \frac{\partial D}{\partial a_{nn}} + \frac{\mu_0 \mu_n}{zz'} \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}}}$$

$$= \left(x \frac{\partial D}{\partial a_{nn}} - x' \frac{\partial D}{\partial a_{n1}}\right) \Big/ \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}},$$

when z, z' are made zero.

So also

$$\mu_n \frac{x' - \xi_n}{z'} = \left(-x \frac{\partial D}{\partial a_{1n}} + x' \frac{\partial D}{\partial a_{11}}\right) \Big/ \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}},$$

and, if for the corresponding equations in η, we use b instead of a and Δ for D, then

$$V = \{\mu_0 z + \mu_1 \tau_1 + \ldots + \mu_n z'\}$$

$$+ \tfrac{1}{2} \left\{ \left(x^2 \frac{\partial D}{\partial a_{nn}} - 2xx' \frac{\partial D}{\partial a_{1n}} + x'^2 \frac{\partial D}{\partial a_{11}}\right) \Big/ \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}} \right.$$

$$\left. + \left(y^2 \frac{\partial \Delta}{\partial b_{nn}} - 2yy' \frac{\partial \Delta}{\partial b_{1n}} + y'^2 \frac{\partial \Delta}{\partial b_{11}}\right) \Big/ \frac{\partial^2 \Delta}{\partial b_{11} \partial b_{nn}} \right\},$$

and the equations for the direction cosines of the ray give

$$-\mu_0 (ax + hy) = \left(x \frac{\partial D}{\partial a_{nn}} - x' \frac{\partial D}{\partial a_{1n}}\right) \Big/ \frac{\partial^2 D}{\partial a_{11} \partial a_{nn}}$$

$$-\mu_0 (hx + by) = \left(y \frac{\partial \Delta}{\partial b_{nn}} - y' \frac{\partial \Delta}{\partial b_{1n}}\right) \Big/ \frac{\partial^2 \Delta}{\partial b_{11} \partial b_{nn}}.$$

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For simplicity, suppose that the principal planes of the incident pencil are those of reference; then h is zero, and

$$\left(\mu_0 a \frac{\partial^2 D}{\partial a_{11}\partial a_{nn}} + \frac{\partial D}{\partial a_{nn}}\right) x = x' \frac{\partial D}{\partial a_{1n}},$$

$$\left(\mu_0 b \frac{\partial^2 \Delta}{\partial b_{11}\partial b_{nn}} + \frac{\partial \Delta}{\partial b_{nn}}\right) y = y' \frac{\partial \Delta}{\partial b_{1n}},$$

and therefore on substitution V becomes

$$\mu_0\tau_0 + \ldots + \mu_n\tau_n - \frac{\mu_n}{2}(a'x'^2 + b'y'^2)$$

$$+ \tfrac{1}{2}\left\{x'^2 \frac{\partial D}{\partial a_{11}} \Big/ \frac{\partial^2 D}{\partial a_{11}\partial_{nn}} + y'^2 \frac{\partial \Delta}{\partial b_{11}} \Big/ \frac{\partial^2 \Delta}{\partial b_{11}\partial b_{nn}}\right\}$$

$$- \tfrac{1}{2}\left\{\frac{x'^2\left(\frac{\partial D}{\partial a_{1n}}\right)^2}{\frac{\partial^2 D}{\partial a_{11}\partial a_{nn}}\left(\mu_0 a \frac{\partial^2 D}{\partial a_{11}\partial a_{nn}} + \frac{\partial D}{\partial a_{nn}}\right)}\right.$$

$$\left. + \frac{y'^2\left(\frac{\partial \Delta}{\partial b_{1n}}\right)^2}{\frac{\partial^2 \Delta}{\partial b_{11}\partial b_{nn}}\left(\mu_0 b \frac{\partial \Delta}{\partial b_{11}\partial b_{nn}} + \frac{\partial \Delta}{\partial b_{nn}}\right)}\right\},$$

and so

$$\mu_n a' = \frac{1}{\frac{\partial^2 D}{\partial a_{11}\partial a_{nn}}}\left\{\frac{\partial D}{\partial a_{11}} - \frac{\left(\frac{\partial D}{\partial a_{1n}}\right)^2}{\mu_0 a \frac{\partial^2 D}{\partial a_{11}\partial a_{nn}} + \frac{\partial D}{\partial a_{nn}}}\right\}$$

$$= \frac{\mu_0 a \frac{\partial D}{\partial a_{11}} + D}{\mu_0 a \frac{\partial^2 D}{\partial a_{11}\partial a_{nn}} + \frac{\partial D}{\partial a_{nn}}}, \quad \text{since} \begin{vmatrix} \frac{\partial D}{\partial a_{11}}, & \frac{\partial D}{\partial a_{1n}} \\ \frac{\partial D}{\partial a_{n1}}, & \frac{\partial D}{\partial a_{nn}} \end{vmatrix} = D \frac{\partial^2 D}{\partial a_{11}\partial a_{nn}}.$$

These determinants D, $\frac{\partial D}{\partial a_{11}}$, $\frac{\partial D}{\partial a_{nn}}$, $\frac{\partial^2 D}{\partial a_{11}\partial a_{nn}}$, are all of the same type, and D can be easily expanded by a mnemonic formula thus. $D \times \left(\frac{\tau_1}{\mu_1}\frac{\tau_2}{\mu_2} \ldots \frac{\tau_{n-1}}{\mu_{n-1}}\right)$ is the power K of the n refracting surfaces, of curvatures $r_1, r_2, \ldots, r_n$ in the primary plane. Let $\kappa_1 = (\mu_1 - \mu_0)\, r_1$, $\kappa_2 = (\mu_2 - \mu_1)\, r_2$, &c. Then $K = \Sigma\kappa + \Sigma\Sigma\kappa_p\kappa_q\kappa_r \ldots \tau'_{pq}\tau_{qr} \ldots$, where the summation includes all possible products of the powers κ, and the coefficient, *e.g.*

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of any product $\kappa_p \kappa_q \kappa_r$, where $p<q<r$, is (the sum of all the reduced distances τ/μ from the p^{th} to the q^{th} surface) × (sum of all the reduced distances from the q^{th} to the r^{th} surface). It is essential that these reduced distances be taken onwards with the light, so that in them no part of the axis is traversed twice.

There are *in the primary plane* two principal foci F_1 and F_2: the position of the first principal focus is given by putting a' zero, and $-\frac{\mu_0}{D}\frac{\partial D}{\partial a_{11}}$ is the distance of F_1 from O; next putting a zero, $\frac{\mu_n}{D}\frac{\partial D}{\partial a_{nn}}$ is the distance of F_2 from O'. Also if q_1 be the primary focus of any incident pencil, q_1' of the emergent pencil, then it follows from the value of a' that

$$\frac{F_1 q_1}{\mu_0} \cdot \frac{F_2 q_1'}{\mu_n} = -\left\{\frac{\partial D}{\partial a_{1n}} \Big/ D\right\}^2 = -(1/K)^2.$$

Section II. Media of Varying Density.

When we try to determine the successive orthogonal surfaces of a small pencil of rays passing through a heterogeneous medium, we are met by the difficulty that to determine V, it is necessary to know the path of any ray, and this involves the solution of a differential equation of the second order, except in the simple cases of a medium stratified in planes, spheres, or cylinders.

First take the case of a medium in which $\mu = f(z)$. Let AB be the path of the axis of a small pencil, PQ the path of any near ray. Let (ξ, η, ζ) be the coordinates of P relative to A, ξ', η', ζ' those of Q relative to B.

Since $\mu = f(z)$, the path of any ray is given by

$$\mu \frac{dx}{ds} = \text{constant} = c_1, \quad \mu \frac{dy}{ds} = \text{constant} = c_2,$$

and therefore $\mu \frac{dz}{ds} = (\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}}$. Hence by division and integration the path of the ray is the intersection of the two cylinders

$$\left.\begin{aligned} x_Q - x_P &= \int_{z_P}^{z_Q} \frac{c_1 dz}{(\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}}} \\ y_Q - y_P &= \int_{z_P}^{z_Q} \frac{c_2 dz}{(\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

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Also

$$\frac{\partial V}{\partial x} = \mu \frac{dx}{ds} = c_1, \frac{\partial V}{\partial y} = c_2, \frac{\partial V}{\partial z} = (\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}},$$

and therefore

$$V_Q - V_P = c_1(x_Q - x_P) + c_2(y_Q - y_P) + \int_{z_P}^{z_Q} (\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}} dz \ldots (2),$$

$$= \int_{z_P}^{z_Q} \frac{\mu^2 dz}{(\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We are supposed to determine c_1 and c_2 from (1) in terms of the coordinates of P and Q and then to substitute them in (2) or (3). This is not possible, but we can find the value of V for points near AB as follows.

Let (x, y, z) be the coordinates of A, (x', y', z') those of B. Then, by Taylor's theorem,

$$V_Q - V_P = f(x' + \xi', y' + \eta', z' + \xi', x + \xi, y + \eta, z + \xi)$$

$$= V' - V + \left(\xi' \frac{\partial}{\partial x'} + \ldots\right)(V' - V) + \tfrac{1}{2}\left(\xi' \frac{\partial}{\partial x'} + \ldots\right)^2 (V' - V).$$

The first differential coefficients are already found

$$\frac{\partial V'}{\partial x'} = c_1, \frac{\partial V'}{\partial y'} = c_2, \frac{\partial V'}{\partial z'} = (\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}},$$

$$\frac{\partial V}{\partial x} = c_1, \frac{\partial V}{\partial y} = c_2, \frac{\partial V}{\partial z} = (\mu^2 - c_1^2 - c_2^2)^{\frac{1}{2}}.$$

For the second differential coefficients, c_1 and c_2 are functions of x', y', z' determined by equations (1). Differentiate these integrals partially; and let I_1 denote $\int_z^{z'} \frac{(\mu^2 - c_2^2)\, dz}{(\mu^2 - c_1^2 - c_2^2)^{\frac{3}{2}}}$, I_2 denote $\int_z^{z'} \frac{\mu^2 - c_1^2}{(\mu^2 - c_1^2 - c_2^2)^{\frac{3}{2}}} dz$, and I_3 denote $\int_z^{z'} \frac{dz}{(\mu^2 - c_1^2 - c_2^2)^{\frac{3}{2}}}$.

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Then, differentiating (1) with regard to x,

$$\left.\begin{aligned} 1 &= \frac{\partial c_1}{\partial x'} I_1 + c_1 c_2 \frac{\partial c_2}{\partial x'} I_2 \\ 0 &= c_1 c_2 \frac{\partial c_1}{\partial x'} I_2 + \frac{\partial c_2}{\partial x'} I_2 \end{aligned}\right\},$$

with regard to y',

$$\left.\begin{aligned} 0 &= \frac{\partial c_1}{\partial y'} I_1 + c_1 c_2 \frac{\partial c_2}{\partial y'} I_2 \\ 1 &= c_1 c_2 \frac{\partial c_1}{\partial y'} I_2 + \frac{\partial c_2}{\partial y'} I_2 \end{aligned}\right\},$$

and with regard to z',

$$\left.\begin{aligned} 0 &= \frac{c_1}{(\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}}} + \frac{\partial c_1}{\partial z'} I_1 + c_1 c_2 \frac{\partial c_2}{\partial z'} I_2 \\ 0 &= \frac{c_2}{(\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}}} + c_1 c_2 \frac{\partial c_1}{\partial z'} I_2 + \frac{\partial c_2}{\partial z'} I_2 \end{aligned}\right\}.$$

Now let the plane of xz be so chosen that it contains the path AB of the axis; so that for it $c_1 = \mu \sin\phi = \mu' \sin\phi'$, and $c_2 = 0$; hence the values of the above differential coefficients at B are

$$\frac{\partial c_1}{\partial x'} = 1/I_1, \frac{\partial c_1}{\partial y'} = 0, \quad \frac{\partial c_1}{\partial z'} = -\tan\phi'/I_1,$$

$$\frac{\partial c_2}{\partial x'} = 0, \quad \frac{\partial c_2}{\partial y'} = 1/I_2, \frac{\partial c_2}{\partial z'} = 0.$$

In the same way, differentiating with regard to the lower limit, we find that at A

$$\frac{\partial c_1}{\partial x} = -1/I_1, \frac{\partial c_1}{\partial y} = 0, \frac{\partial c_1}{\partial z} = \tan\phi/I_1,$$

$$\frac{\partial c_2}{\partial x} = 0, \frac{\partial c_2}{\partial y} = -1/I_2, \frac{\partial c_2}{\partial z} = 0.$$

The value of I_1 here is

$$\int_A^B \frac{\mu^2 dz}{(\mu^2 - c_1^2)^{\frac{1}{2}}} = \int_A^B \frac{dz}{\mu \cos^2\phi},$$

and of I_2 is

$$\int_A^B \frac{dz}{(\mu^2 - c_1^2)^{\frac{1}{2}}} = \int_A^B \frac{dz}{\mu \cos\phi}.$$

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Hence we easily find for B

$$\left.\begin{aligned} &\frac{\partial^2 V'}{\partial x'^2} = \frac{\partial c_1}{\partial x'} = \frac{1}{I_1}, \quad \frac{\partial^2 V'}{\partial y'^2} = \frac{\partial c_2}{\partial y'} = \frac{1}{I_2}, \quad \frac{\partial^2 V'}{\partial x' \partial y'} = \frac{\partial c_2}{\partial x'} = 0 \\ &\frac{\partial^2 V'}{\partial x' \partial z'} = \frac{\partial c_1}{\partial z'} = -\frac{\tan\phi'}{I_1}, \quad \frac{\partial^2 V'}{\partial y' \partial z'} = \frac{\partial c_2}{\partial z'} = 0 \\ \text{and } &\frac{\partial^2 V'}{\partial z'^2} = \left(\mu' \frac{d\mu'}{dz'} - c_1 \frac{\partial c_1}{\partial z'} - c_2 \frac{\partial c_2}{\partial z'}\right)(\mu'^2 - c_1^2 - c_2^2)^{-\frac{1}{2}} \\ &\qquad = \frac{1}{\cos\phi'} \frac{d\mu'}{dz'} + \frac{\tan^2\phi'}{I_1} \end{aligned}\right\}.$$

So also for A,

$$\left.\begin{aligned} &\frac{\partial^2 V}{\partial x^2} = \frac{\partial c_1}{\partial x} = -\frac{1}{I_1}, \quad \frac{\partial^2 V}{\partial y^2} = -\frac{1}{I_2}, \quad \frac{\partial^2 V}{\partial x \partial y} = 0 \\ &\frac{\partial^2 V}{\partial x \partial z} = \frac{\partial c_1}{\partial z} = +\frac{\tan\phi}{I_1}, \quad \frac{\partial^2 V}{\partial y \partial z} = 0 \\ \text{and } &\frac{\partial^2 V}{\partial z^2} = \frac{1}{\cos\phi} \frac{d\mu}{dz} - \frac{\tan^2\phi}{I_1} \end{aligned}\right\}.$$

For the other product terms,

$$\left.\begin{aligned} &\frac{\partial^2}{\partial x \partial x'}(V' - V) = \frac{\partial c_1}{\partial x} = -\frac{1}{I_1}, \quad \frac{\partial^2}{\partial x \partial y'}(V' - V) = \frac{\partial c_2}{\partial x} = 0 \\ &\frac{\partial^2}{\partial x \partial z'}(V' - V) = \frac{\partial}{\partial x}(\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}} = \frac{-c_1 \dfrac{\partial c_1}{\partial x}}{\mu' \cos\phi'} = \frac{\tan\phi'}{I_1} \end{aligned}\right\},$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial y \partial x'}(V' - V) = \frac{\partial c_1}{\partial y} = 0, \quad \frac{\partial^2}{\partial y \partial y'}(V' - V) = \frac{\partial c_2}{\partial y} = -\frac{1}{I_2} \\ &\frac{\partial^2}{\partial y \partial z'}(V' - V) = \frac{\partial}{\partial y}(\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}} = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial z \partial x'}(V' - V) = \frac{\partial c_1}{\partial z} = \frac{\tan\phi}{I_1}, \quad \frac{\partial^2}{\partial z \partial y'}(V' - V) = \frac{\partial c_2}{\partial z} = 0 \\ &\frac{\partial^2}{\partial z \partial z'}(V' - V) = \frac{\partial}{\partial z}(\mu'^2 - c_1^2 - c_2^2)^{\frac{1}{2}} = \frac{-c_1 \dfrac{\partial c_1}{\partial z}}{\mu' \cos\phi'} = \frac{-\tan\phi' \tan\phi}{I_1} \end{aligned}\right\}.$$

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Substitute these values in the expansion above, and we have

$$V_Q - V_P = V' - V + \mu'(\xi' \sin\phi' + \zeta' \cos\phi') - \mu(\xi \sin\phi + \zeta \cos\phi)$$
$$+ \tfrac{1}{2}\left[\frac{\{(\xi' - \zeta' \tan\phi') - (\xi - \zeta \tan\phi)\}^2}{I_1}\right.$$
$$\left. + \frac{(\eta' - \eta)^2}{I_2} + \frac{\zeta'^2}{\cos\phi'}\frac{d\mu'}{dz'} - \frac{\zeta^2}{\cos\phi}\frac{d\mu}{dz}\right] \quad \ldots\ldots\ldots(4).$$

We cannot here determine the relation between the points Q and P by putting $\frac{\partial V}{\partial \xi}$, &c. zero in the ordinary way, for we have already assumed that PQ is the path of a ray in determining this value of V.

But let the equation of the orthotomic surface at A be $Z = \frac{1}{2}(aX^2 + 2hX\eta + b\eta^2)$ referred to axes of reference, of which Z is the axis of the pencil and ZX is the plane of incidence of the axis. Putting in this $Z = \zeta \cos\phi + \xi \sin\phi$, $X = \xi \cos\phi - \zeta \sin\phi$, the equation of the orthotomic surface is

$$\zeta \cos\phi + \xi \sin\phi$$
$$= \tfrac{1}{2}\{a(\xi \cos\phi - \zeta \sin\phi)^2 + 2h\eta(\xi \cos\phi - \zeta \sin\phi) + b\eta^2\} \ldots(5),$$

and the direction cosines of the normal are, neglecting squares and products of ξ, η, ζ,

$$l = \sin\phi - \{a \cos\phi(\xi \cos\phi - \zeta \sin\phi) + h\eta \cos\phi\},$$
$$m = -\{h(\xi \cos\phi - \zeta \sin\phi) + b\eta\},$$
$$n = \cos\phi + \{a \sin\phi(\xi \cos\phi - \zeta \sin\phi) + h\eta \sin\phi).$$

Hence at P the value of c_1, or μl, differs from its value at A by $\mu\delta l + \zeta \frac{d\mu}{dz} \sin\phi$; that is,

$$\delta c_1 = \zeta \sin\phi \frac{d\mu}{dz} - \mu \cos\phi\{a(\xi \cos\phi - \zeta \sin\phi) + h\eta\}.$$

So also

$$\delta c_2 = -\mu\{h(\xi \cos\phi - \zeta \sin\phi) + b\eta\}.$$

But taking equations (1) and substituting $x' + \xi'$ for x_Q, $z' + \zeta'$ for z_Q, and $c_1 + \delta c_1$, δc_2 for c_1 and c_2, we have

$$\xi' - \xi = \zeta' \tan\phi' - \zeta \tan\phi + \delta c_1 . I_1,$$
$$\eta' - \eta = \qquad\qquad\qquad \delta c_2 . I_2.$$

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Hence

$$\left.\begin{aligned}\xi' - \zeta \tan\phi' &= (\xi - \zeta\tan\phi)(1 - I_1\mu a\cos^2\phi)\\ &\quad - \eta I_1\mu h\cos\phi + \zeta I_1 \sin\phi\frac{d\mu}{dz}\\ \eta' &= \eta(1-\mu b I_2) - (\xi - \zeta\tan\phi) I_2\mu h\cos\phi\end{aligned}\right\}(6).$$

These equations are the conditions that a ray incident at P may pass through Q.

Lastly, let the equation of the orthotomic surface through B be

$$\zeta'\cos\phi' + \zeta'\sin\phi' = \tfrac{1}{2}\{a'(\xi'\cos\phi' - \zeta'\sin\phi')^2 + 2h'\eta'(\xi'\cos\phi' - \zeta\sin\phi') + b'\eta'^2\} \quad ...(7).$$

Then if from equations (5), (7), and (6) we substitute in (4) for ξ, η, ζ, ζ' in terms of ξ' and η', the variable part must vanish identically.

By (7) it follows that $\zeta'\cos\phi' + \xi'\sin\phi$ is a quadratic function of ξ', η', ζ', and therefore

$$\zeta' = -\sin\phi'\cos\phi'(\xi' - \zeta'\tan\phi') + \cos\phi'(\text{quad. function}).$$

Denote $\xi' - \zeta'\tan\phi'$ by χ', then $\zeta' = -\chi'\sin\phi'\cos\phi'$ approximately, and we can write (4) as

$$\mu'[a'\chi'^2\cos^2\phi' + 2h'\chi'\eta'\cos\phi' + b'\eta'^2] - \mu[a\chi^2\cos^2\phi + 2h\chi\eta\cos\phi + b\eta^2] + \left[\frac{(\chi'-\chi)^2}{I_1} + \frac{(\eta'-\eta)^2}{I_2} + \sin^2\phi'\cos\phi'\frac{d\mu'}{dz'}\chi'^2 - \sin^2\phi\cos\phi\frac{d\mu}{dz}\chi^2\right] \equiv 0 \quad .....(8),$$

and (6) as

$$\chi = \chi\left(1 - I_1\mu a\cos^2\phi - I_1\sin^2\phi\cos\phi\frac{d\mu}{dz}\right) - \eta I_1\mu h\cos\phi,$$

$$\eta' = -\chi I_2\mu h\cos\phi + \eta(1-\mu b I_2) \quad ..... \quad .....(9).$$

From (9), we obtain

$$\left.\begin{aligned}\chi &= -\chi'(\mu b I_2 - 1) + \eta' I_1\mu h\cos\phi\\ \eta &= +\chi' I_2\mu h\cos\phi - \eta'\left(I_1\mu a\cos^2\phi - 1 + I_1\sin^2\phi\cos\phi\frac{d\mu}{dz}\right)\end{aligned}\right\} \div D,$$

where $D = \begin{vmatrix} I_1\mu a\cos^2\phi - 1 + I_1\sin^2\phi\cos\phi\frac{d\mu}{dz}, & I_1\mu h\cos\phi \\ I_2\mu h\cos\phi, & \mu b I_2 - 1\end{vmatrix}.$

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Again, equation (8) is of the form

$$\left(\mu' a' \cos^2\phi' + \frac{1}{I_1} + \sin^2\phi' \cos\phi' \frac{d\mu'}{dz'}\right) \chi'^2 + 2\mu' h' \chi' \eta' \cos\phi' + \left(\mu' b' + \frac{1}{I_2}\right) \eta'^2 - 2\left(\frac{\chi\chi'}{I_1} + \frac{\eta\eta'}{I_2}\right) + (\alpha\chi^2 + 2\gamma\chi\eta + \beta\eta^2) = 0,$$

where $\chi' = I_1(\alpha\chi + \gamma\eta)$, $\eta' = I_2(\gamma\chi + \beta\eta)$,

and therefore

$$\alpha\chi^2 + 2\gamma\chi\eta + \beta\eta^2 = \frac{\chi\chi'}{I_1} + \frac{\eta\eta'}{I_2}.$$

Hence the equations

$$\left.\begin{aligned} \mu' a' \cos^2\phi' + \frac{1}{I_1} + \sin^2\phi' \cos\phi' \frac{d\mu'}{dz'} + \frac{\mu b I_2 - 1}{I_1 D} &= 0 \\ \mu' h' \cos\phi' = \frac{\mu h \cos\phi}{D} & \\ \mu' b' + \frac{1}{I_2} + \frac{I_1 \mu a \cos^2\phi - 1 + I_1 \sin^2\phi \cos\phi \frac{d\mu}{dz}}{I_2 D} &= 0 \end{aligned}\right\},$$

give the constants, a', h', b', of the orthotomic surface at any point.

In the simple case where the primary focal plane of the pencil is the plane of incidence h is zero, and these equations can be simplified.

The equation for the primary focus is

$$\mu' a' \cos^2\phi' + \frac{1}{I_1} + \sin^2\phi' \cos\phi' \frac{d\mu'}{dz'} + \frac{1}{I_1^2 \left\{\mu a \cos^2\phi - \frac{1}{I_1} + \sin^2\phi \cos\phi \frac{d\mu}{dz}\right\}} = 0$$

which can be reduced to

$$\left[\mu a \cos^2\phi + \sin^2\phi \cos\phi \frac{d\mu}{dz}\right]^{-1} - \left[\mu' a' \cos^2\phi' + \sin^2\phi' \cos\phi' \frac{d\mu'}{dz'}\right]^{-1} = I_1 = \int_z^{z'} \frac{dz}{\mu \cos^2\phi},$$

or

$$\left[\mu a \cos^2\phi + \sin^2\phi \cos\phi \frac{d\mu}{dz}\right]^{-1} + \int^z \frac{dz}{\mu \cos^2\phi}$$

is a constant for this pencil.

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The equation for the secondary focus is

$$\mu' b' + \frac{1}{I_2} + \frac{1}{I_2(\mu b I_2 - 1)} = 0,$$

which is equivalent to

$$1/\mu b - 1/\mu' b' = I_2 = \int_z^{z'} \frac{dz}{\mu \cos\phi},$$

or $\dfrac{1}{\mu b} + \displaystyle\int^z \frac{dz}{\mu \cos\phi}$ is constant, which only expresses that the secondary focus always lies on the same normal to the plane refracting surfaces.

Secondly, let the media be stratified in spheres; and let the path of any ray be referred to polar coordinates r, θ, ψ. We know that any ray lies in a diametral plane, and that its equation is given by $\mu p = \mu r \sin\phi = \text{constant}$. This is

$$\mu^2 r^2 \left\{ \left(r \frac{d\theta}{ds} \right)^2 + \left(r \sin\theta \frac{d\psi}{ds} \right)^2 \right\} = c_1^2,$$

or

$$\mu^2 r^2 \{ (ds)^2 - (dr)^2 \} = c_1^2 (ds)^2.$$

Again, the path is that of a particle moving with velocity μ under a central force, and therefore the equation of angular momentum about the axis of z gives

$$\mu r \sin\theta \frac{d\psi}{ds} r \sin\theta = c_2.$$

Substituting above for $\mu r^2 \sin\theta \dfrac{d\psi}{ds}$, we have

$$\mu^2 r^4 \left(\frac{d\theta}{ds} \right)^2 + \frac{c_2^2}{\sin^2\theta} = c_1^2.$$

Hence

$$\mu^2 r^4 (d\theta)^2 = \frac{c_1^2 \sin^2\theta - c_2^2}{\sin^2\theta} (ds)^2 = \frac{c_1^2 \sin^2\theta - c_2^2}{\sin^2\theta} \cdot \frac{\mu^2 r^2 (dr)^2}{\mu^2 r^2 - c_1^2},$$

therefore one equation for the path is

$$\int_{\theta_P}^{\theta_Q} \frac{\sin\theta \, d\theta}{(c_1^2 \sin^2\theta - c_2^2)^{\frac{1}{2}}} = \int_{r_P}^{r_Q} \frac{dr}{r (\mu^2 r^2 - c_1^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots(1).$$

The relation between ψ and θ is obtained from

$$\mu^2 r^4 (d\theta)^2 = \frac{c_1^2 \sin^2\theta - c_2^2}{\sin^2\theta} (ds)^2 = (c_1^2 \sin^2\theta - c_2^2) \frac{\mu^2 r^4 \sin^2\theta (d\psi)^2}{c_2^2},$$

or

$$\psi_Q - \psi_P = \int_{\theta_P}^{\theta_Q} \frac{c_2 \, d\theta}{\sin\theta (c_1^2 \sin^2\theta - c_2^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots(2).$$

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Again $$\frac{\partial V}{\partial r} = \mu \frac{dr}{ds} = \frac{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}}{r},$$

$$\frac{\partial V}{r\partial\theta} = \mu r \frac{d\theta}{ds} = \frac{(c_1^2 \sin^2\theta - c_2^2)^{\frac{1}{2}}}{r \sin\theta},$$

$$\frac{\partial V}{r \sin\theta \partial\psi} = \mu r \sin\theta \frac{d\psi}{ds} = \frac{c_3}{r \sin\theta},$$

so that

$$V_Q - V_P = \int_{r_P}^{r_Q} \frac{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}}{r} dr + \int_{\theta_P}^{\theta_Q} \frac{(c_1^2 \sin^2\theta - c_2^2)^{\frac{1}{2}}}{\sin\theta} d\theta + c_3 (\psi_Q - \psi_P) \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$= \int_{r_P}^{r_Q} \frac{\mu^2 r\, dr}{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now following the notation of the previous example, let AB be the path of the axis, and let it lie in the plane $\psi = 0$, so that for it $c_3 = 0$, and r, θ are the plane polar coordinates. Let A be the origin for rectangular coordinates ξ, η, ζ, of which ζ is along the normal r, and the plane $\zeta\xi$ is the plane of incidence; and let B be the origin of similar coordinates ξ', η', ζ'.

Then, as before, we may expand $V_Q - V_P$ by Taylor's theorem in terms of the small quantities $\xi, \ldots, \zeta'$. If u stand for $V' - V$, then $\frac{\partial V}{\partial \xi} = \frac{\partial V}{r\partial\theta}$, $\frac{\partial V}{\partial \eta} = \frac{\partial V}{r \sin\theta \partial\psi}$, $\frac{\partial V}{\partial \zeta} = \frac{\partial V}{\partial r}$; but similar equations do not hold for the second differentials. By the method of moving axes, it is easy to shew that we must substitute as follows:—

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{r^2 \partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad \frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{\partial^2 u}{r^2 \sin\theta \partial\theta\partial\psi} - \frac{\cos\theta}{r^2 \sin^2\theta}\frac{\partial u}{\partial\psi},$$

$$\frac{\partial^2 u}{\partial\xi\partial\zeta} = \frac{\partial^2 u}{r\partial\theta\partial r} - \frac{\partial u}{r^2\partial\theta},$$

$$\frac{\partial^2 u}{\partial\eta^2} = \frac{\partial^2 u}{r^2 \sin^2\theta \partial\psi^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial u}{\partial\theta},$$

$$\frac{\partial^2 u}{\partial\eta\partial\zeta} = \frac{\partial^2 u}{r \sin\theta \partial\psi \partial r} - \frac{\partial u}{r^2 \sin\theta \partial\psi}, \quad \frac{\partial^2 u}{\partial\zeta^2} = \frac{\partial^2 u}{\partial r^2}, \quad \ldots\ldots\ldots\ldots(5).$$

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Similarly with dashed letters for $\frac{\partial^2 u}{\partial \xi^2}$, &c.; but for $\frac{\partial^2 u}{\partial \xi \partial \xi'}$ and all such products no differential coefficients of the first degree occur, since the rotations of the axes at A do not affect those at B and conversely.

For the first differential coefficients

$$\frac{\partial u}{\partial r} = \frac{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}}{r} = \mu \cos\phi\,; \quad \frac{\partial u}{\partial r'} = -\mu\cos\phi',$$

$$\frac{\partial u}{r\partial\theta} = \frac{(c_1^2\sin^2\theta - c_2^2)^{\frac{1}{2}}}{r\sin\theta} = \mu\sin\phi\,, \quad \frac{\partial u}{\partial r'} = -\mu\sin\phi',$$

$$\frac{\partial u}{r\sin\theta\,\partial\psi} = \frac{c_2}{r\sin\theta} = 0, \quad \frac{\partial u}{r\sin\theta'\partial\psi'} = 0,$$

for the axis of the pencil.

Again, to determine the values of the partial differential coefficients of c_1 and c_2, we have on differentiating equations (1) and (2) and making c_2 zero after differentiation

$$\frac{\partial\theta' - \partial\theta}{c_1} - \frac{\partial c_2}{c_1^2}(\theta' - \theta) \equiv \frac{\partial r'}{r'(\mu^2 r'^2 - c_1^2)^{\frac{1}{2}}} - \frac{\partial r}{r(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}} + c_1\partial c_1\int_r^{r'} \frac{dr}{r(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}},$$

or if we substitute in this $\frac{\theta' - \theta}{c_1} = \int_r^{r'} \frac{dr}{r(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}}$,

$$\partial\theta' - \partial\theta = \frac{c_1\partial r'}{r'(\mu^2 r'^2 - c_1^2)^{\frac{1}{2}}} - \frac{c_1\partial r}{r(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}} + \partial c_1\int_r^{r'} \frac{\mu^2 r\,dr}{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}},$$

and $$\partial\psi' - \partial\psi = \partial c_2\int_\theta^{\theta'} \frac{c_1^2\sin\theta\,d\theta}{(c_1^2\sin^2\theta - c_2^2)^{\frac{1}{2}}} + c_2\,(\text{terms})$$

$$= \partial c_2\,\frac{\cot\theta - \cot\theta'}{c_1}, \text{ when } c_2 = 0.$$

Let I denote

$$\int_r^{r'} \frac{\mu^2 r\,dr}{(\mu^2 r^2 - c_1^2)^{\frac{1}{2}}} = \int_r^{r'} \frac{dr}{\mu r'\cos^2\phi},$$

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then

$$\left.\begin{aligned} \frac{\partial c_1}{\partial \theta'} &= \frac{1}{I}, \ \frac{\partial c_1}{\partial \psi'} = 0, \ \frac{\partial c_1}{\partial r'} = -\frac{\tan\phi'}{r'I} \\ \frac{\partial c_2}{\partial \theta'} &= 0, \ \frac{\partial c_2}{\partial \psi'} = \frac{c_1 \sin\theta \sin\theta'}{\sin(\theta'-\theta)}, \ \frac{\partial c_2}{\partial r'} = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial c_1}{\partial \theta} &= -\frac{1}{I}, \ \frac{\partial c_1}{\partial \psi} = 0, \ \frac{\partial c_1}{\partial r'} = \frac{\tan\phi}{rI} \\ \frac{\partial c_2}{\partial \theta} &= 0, \ \frac{\partial c_2}{\partial \psi} = \frac{-c_1 \sin\theta \sin\theta'}{\sin(\theta'-\theta)}, \ \frac{\partial c_2}{\partial r} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6).$$

With these values

$$\frac{\partial^2 u}{\partial r'^2} = \frac{r'\mu' \dfrac{d\mu'}{dr'}}{(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} + \frac{c_1^2}{r'^2(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} - \frac{c_1}{r'(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} \frac{\partial c_1}{\partial r'}$$

$$= \frac{1}{\cos\phi'} \frac{d\mu'}{dr'} + \frac{\mu'}{r'} \frac{\sin^2\phi'}{\cos\phi'} + \frac{\tan^2\phi'}{r'^2 I},$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{\cos\phi} \frac{d\mu}{dr} - \frac{\mu}{r} \frac{\sin^2\phi}{\cos\phi} + \frac{\tan^2\phi}{r^2 I},$$

$$\frac{\partial^2 u}{\partial r \partial r'} = -\frac{c_1 \dfrac{\partial c_1}{\partial r}}{r'(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} = -\frac{\tan\phi \tan\phi'}{rr'I},$$

$$\frac{\partial^2 u}{\partial r' \partial \theta'} = -\frac{c_1 \dfrac{\partial c_1}{\partial \theta'}}{r'(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} = -\frac{\tan\phi'}{r'I},$$

$$\frac{\partial^2 u}{\partial r \partial \theta} = -\frac{\tan\phi}{rI}, \ \frac{\partial^2 u}{\partial \theta \partial r'} = \frac{-c_1 \dfrac{\partial c_1}{\partial \theta}}{r'(\mu'^2 r'^2 - c_1^2)^{\frac{1}{2}}} = +\frac{\tan\phi'}{r'I},$$

$$\frac{\partial^2 u}{\partial \theta' \partial r} = +\frac{\tan\phi}{rI}; \ \frac{\partial^2 u}{\partial r' \partial \psi'} = \frac{\partial^2 u}{\partial r \partial \psi} = \frac{\partial^2 u}{\partial r \partial \psi'} = \frac{\partial^2 u}{\partial r' \partial \psi} = 0.$$

So from $\dfrac{\partial u}{\partial \theta'} = \dfrac{(c_1^2 \sin^2\theta' - c_2^2)^{\frac{1}{2}}}{\sin\theta'} = c_1$, when $c_2 = 0$, we deduce

$$\frac{\partial^2 u}{\partial \theta'^2} = \frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial^2 u}{\partial \theta \partial \theta'} = \frac{1}{I},$$

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$$\frac{\partial^2 u}{\partial \vartheta' \partial \psi'} = \frac{\partial^2 u}{\partial \vartheta \partial \psi} = \frac{\partial^2 u}{\partial \vartheta' \partial \psi} = \frac{\partial^2 u}{\partial \vartheta \partial \psi'} = 0,$$

and from $\frac{\partial u}{\partial \psi'} = c_2$ we deduce

$$\frac{\partial^2 u}{\partial \psi'^2} = c_1 \frac{\sin\theta \sin\theta'}{\sin(\theta' - \theta)} = \frac{\partial^2 u}{\partial \psi^2} = -\frac{\partial^2 u}{\partial \psi \partial \psi'}.$$

Substitute these values in (5); we obtain

$$\frac{\partial^2 u}{\partial \xi'^2} = \frac{1}{r'^2 I} + \frac{\mu' \cos\phi'}{r'}, \quad \frac{\partial^2 u}{\partial \xi^2} = \frac{1}{r^2 I} - \frac{\mu \cos\phi}{r},$$

$$\frac{\partial^2 u}{\partial \xi \partial \xi'} = -\frac{1}{rr' I};$$

$$\frac{\partial^2 u}{\partial \xi' \partial \eta'} = \&c. = 0,$$

$$\frac{\partial^2 u}{\partial \xi' \partial \zeta'} = -\frac{\tan\phi'}{r'^2 I} - \frac{\mu' \sin\phi'}{r'}, \quad \frac{\partial^2 u}{\partial \xi \partial \zeta} = -\frac{\tan\phi}{r^2 I} + \frac{\mu \sin\phi}{r},$$

$$\frac{\partial^2 u}{\partial \xi' \partial \zeta} = \frac{\tan\phi}{rr' I}, \quad \frac{\partial^2 u}{\partial \xi \partial \zeta'} = \frac{\tan\phi'}{rr' I};$$

$$\frac{\partial^2 u}{\partial \eta'^2} = \frac{1}{r'^2 \sin^2\theta'} \frac{c_1 \sin\theta \sin\theta'}{\sin(\theta' - \theta)} + \frac{c_1 \cos\theta'}{r'^2 \sin\theta'} + \frac{\mu' \cos\phi'}{r'}$$

$$= \frac{c_1 \cot\delta}{r'^2} + \frac{\mu' \cos\phi'}{r'}; \text{ where } \delta = \theta' - \theta.$$

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{c_1 \cot\delta}{r^2} - \frac{\mu \cos\phi}{r}, \quad \frac{\partial^2 u}{\partial \eta \partial \eta'} = -\frac{c_1}{rr' \sin\delta},$$

$$\frac{\partial^2 u}{\partial \eta' \partial \zeta'} = \&c. = 0,$$

$$\frac{\partial^2 u}{\partial \zeta'^2} = \frac{1}{\cos\phi'} \frac{d\mu'}{dr'} + \frac{\mu'}{r'} \frac{\sin^2\phi'}{\cos\phi'} + \frac{\tan^2\phi'}{r'^2 I},$$

$$\frac{\partial^2 u}{\partial \zeta^2} = -\frac{1}{\cos\phi} \frac{d\mu}{dr} - \frac{\mu}{r} \frac{\sin^2\phi}{\cos\phi} + \frac{\tan^2\phi}{rr' I}$$

$$\frac{\partial^2 u}{\partial \zeta \partial \zeta'} = -\frac{\tan\phi \tan\phi'}{rr' I}.$$

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Therefore the expansion by Taylor's theorem is

$$V_Q - V_P = V' - V + \mu'(\zeta' \cos\phi' + \xi' \sin\phi') - \mu(\zeta \cos\phi + \xi \sin\phi)$$
$$+ \tfrac{1}{2}\left[\left(\frac{\xi' - \zeta' \tan\phi'}{r'} - \frac{\xi - \zeta \tan\phi}{r}\right)^2 \frac{1}{I}\right.$$
$$+ \frac{\mu' \cos\phi'}{r'}(\xi' - \zeta' \tan\phi')^2 - \frac{\mu \cos\phi}{r}(\xi - \zeta \tan\phi)^2$$
$$+ \frac{\zeta'^2}{\cos\phi'}\frac{d\mu'}{dr'} - \frac{\zeta^2}{\cos\phi}\frac{d\mu}{dr}$$
$$+ \eta'^2\left\{c_1 \frac{\cot\delta}{r'^2} + \frac{\mu' \cos\phi'}{r'}\right\} - \frac{2\eta\eta' c_1}{rr' \sin\delta}$$
$$\left. + \eta^2\left\{\frac{c_1 \cot\delta}{r^2} - \frac{\mu \cos\phi}{r}\right\}\right] \ldots\ldots\ldots\ldots\ldots (7).$$

We have next to express the relation between ξ', η', ζ' and ξ, η, ζ in terms of the constants a, h, b of the orthotomic surface. The equation of this surface referred to the axes through A is

$$\zeta \cos\phi + \xi \sin\phi = \tfrac{1}{2}[a(\xi \cos\phi - \zeta \sin\phi)^2$$
$$+ 2h\eta(\xi \cos\phi - \zeta \sin\phi) + b\eta^2],$$

and the direction cosines of the normal, the ray through P, are

$$l = \sin\phi - \{a \cos\phi(\xi \cos\phi - \zeta \sin\phi) + h\eta \cos\phi\},$$
$$m = -\{h(\xi \cos\phi - \zeta \sin\phi) + b\eta\},$$
$$n = \cos\phi + \sin\phi\{a(\xi \cos\phi - \zeta \sin\phi) + h\eta\}.$$

If p be the perpendicular on this line from the centre of the spheres, coordinates $(0, 0, -r)$,

$$p^2 = \xi^2 + \eta^2 + (r + \zeta)^2 - \{l\xi + m\eta + n(r + \zeta)\}^2$$
$$= r^2(1 - n^2) + 2r\zeta - 2nr(l\xi + n\zeta), \text{ neglecting squares of } \xi, \eta, \zeta$$
$$= r^2[\sin^2\phi - 2 \cos\phi \sin\phi\{a(\xi \cos\phi - \zeta \sin\phi) + h\eta\}]$$
$$+ 2r\zeta - 2r \cos\phi(\xi \sin\phi + \zeta \cos\phi)$$
$$= r^2 \sin^2\phi - 2(r^2 a \cos^2\phi \sin\phi + r \sin\phi \cos\phi)(\xi - \zeta \tan\phi)$$
$$- 2r^2 \cos\phi \sin\phi\, h\eta,$$

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while the perpendicular on the axial ray is $r \sin\phi$, so that

$$p\partial p = -\{(ra\cos\phi + 1)(\xi - \zeta\tan\phi) + rh\eta\}\, r\cos\phi\sin\phi,$$

and

$$\frac{\partial p}{p} = -\{(ra\cos\phi + 1)(\xi - \zeta\tan\phi) + rh\eta\}\,\frac{\cos\phi}{r\sin\phi}.$$

But $\qquad c_1^2 = \mu^2 p^2, \quad \dfrac{\partial c_1}{c_1} = \dfrac{\partial\mu}{\mu} + \dfrac{\partial p}{p},$

i.e. $$\frac{\partial c_1}{c_1} = \frac{\zeta}{\mu}\frac{d\mu}{dr} - \{(ra\cos\phi + 1)(\xi - \zeta\tan\phi) + rh\eta\}\frac{\cos\phi}{r\sin\phi},$$

or $$\partial c_1 = r\sin\phi\frac{d\mu}{dr}\zeta - \mu\cos\phi\{(ra\cos\phi + 1)(\xi - \zeta\tan\phi) + rh\eta\}.$$

But we have the equation

$$\partial\theta' - \partial\theta = \frac{\tan\phi'}{r'}\partial r' - \frac{\tan\phi}{r}\partial r + \partial c_1 I,$$

or $$\frac{\xi' - \zeta'\tan\phi'}{r'} = \frac{\xi - \zeta\tan\phi}{r} + \partial c_1 I$$

$$= \frac{\xi - \zeta\tan\phi}{r}\{1 - I\mu r\cos\phi\,(ra\cos\phi + 1)\} - Irh\eta\,\mu\cos\phi$$

$$+ Ir\sin\phi\frac{d\mu}{dr}\zeta \quad \ldots\ldots\ldots\ldots(8).$$

The other relation is obtained by expressing that the ray PQ lies throughout in a diametral plane.

The equation to this plane is

$$\begin{vmatrix} x, & y, & z \\ r\sin\theta + \xi\cos\theta + \zeta\sin\theta, & \eta, & r\cos\theta - \xi\sin\theta + \zeta\cos\theta \\ l\cos\theta + n\sin\theta & , m, & -l\sin\theta + n\cos\theta \end{vmatrix} = 0.$$

Substitute the coordinates of Q referred to the fixed axes Ox, Oy, Oz, which are

$$r'\sin\theta' + \xi'\cos\theta' + \zeta'\sin\theta',\ \eta',\ r'\cos\theta' - \xi'\sin\theta' + \zeta'\cos\theta';$$

we obtain, neglecting products and squares ξ, η, ζ,

$$(l\cos\theta + n\sin\theta)(\eta' r\cos\theta - \eta r'\cos\theta') - mrr'\sin\delta$$

$$+ (-l\sin\theta + n\cos\theta)(\eta r'\sin\theta' - \eta' r\sin\theta) = 0,$$

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$$i.e. \qquad l\left\{\frac{\eta'}{r'}-\frac{\eta}{r}\cos\delta\right\}-m\sin\delta+n\frac{\eta}{r}\sin\delta=0,$$

$$\text{or } \frac{\eta'}{r'}=\frac{\eta}{r}(\cos\delta-\cot\phi\sin\delta)+\frac{m\sin\delta}{\sin\phi}$$

$$=-h\sin\delta\cot\phi\,(\xi-\zeta\tan\phi)+\frac{\eta}{r}\left\{\cos\delta-\cot\phi\sin\delta-rb\frac{\sin\delta}{\sin\phi}\right\}$$

.....(9).

Proceeding as before, we substitute in (7) and (8) the simplification $\zeta'=-\cos\phi'\sin\phi'\chi'$ + squares, and obtain

$$\frac{\chi'}{r'}=-Ir\chi\left\{\mu a\cos^2\phi+\frac{\mu\cos\phi}{r}+\sin^2\phi\cos\phi\frac{d\mu}{dr}-\frac{1}{r^3I}\right\}-\eta Ih\mu r\cos\phi,$$

$$\frac{\eta'}{r'}=-\chi h\sin\delta\cot\phi-\eta\left\{\frac{b\sin\delta}{\sin\phi}-\frac{\cos\delta-\cot\phi\sin\delta}{r}\right\}\ldots.(10).$$

Moreover it will be found that by the help of these equations, of the equation of the orthotomic surface and of Euler's theorem, equation (7) can be written as

$$2\{V_Q-V_P-(V'-V)\}$$

$$=\chi'^2\left\{\mu'a'\cos^2\phi'+\frac{\mu'\cos\phi'}{r'}+\sin^2\phi'\cos\phi'\frac{d\mu'}{dr'}-\frac{1}{r'^3I}\right\}$$

$$+2\mu'h'\cos\phi'\chi'\eta'+\left\{\mu'b'+\frac{\mu'(\sin\phi'\cot\delta+\cos\phi')}{r'}\right\}\eta'^2$$

$$-\frac{\chi\chi'}{Irr'}-\frac{c_1\eta\eta'}{rr'\sin\delta}.$$

This expression is identically zero for values of χ, η, χ', η' connected by equations (10).

Hence

$$\mu'a'\cos^2\phi'+\frac{\mu'\cos\phi'}{r'}+\sin^2\phi'\cos\phi'\frac{d\mu'}{dr'}+\frac{1}{r'^3I}$$

$$+\frac{1}{DI^2r^2r'^2}\left\{\mu b-\frac{\mu\sin(\phi-\delta)}{r\sin\delta}\right\}=0,$$

$$\mu'b'+\frac{\mu'\sin(\phi'+\delta)}{r'\sin\delta}$$

$$+\frac{c_1^2}{Dr^2r'^2\sin^2\delta}\left\{\mu a\cos^2\phi+\frac{\mu\cos\phi}{r}+\sin^2\phi\cos\phi\frac{d\mu}{dr}-\frac{1}{r^3I}\right\}=0,$$

$$\mu'h'\cos\phi'=\mu h\cos\phi\frac{c_1}{DIr^2r'^2\sin\delta},$$

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where

$$D = \begin{vmatrix} \mu a \cos^2\phi + \dfrac{\mu \cos\phi}{r} + \sin^2\phi \cos\phi \dfrac{d\mu}{dr} - \dfrac{1}{r^2 I}, & \mu h \cos\phi \\ \mu h \cos\phi, & \mu b - \dfrac{\mu \sin(\phi - \delta)}{r \sin\delta} \end{vmatrix}.$$

Let the plane of incidence be the primary focal plane, $h' = h = 0$: then

$$\mu' a' \cos^2\phi' + \frac{\mu' \cos\phi'}{r'} + \sin^2\phi' \cos\phi' \frac{d\mu}{dr'} + \frac{1}{r'^2 I}$$

$$+ \left[I^2 r^2 r'^2 \left(\mu a \cos^2\phi + \frac{\mu \cos\phi}{r} + \sin^2\phi \cos\phi \frac{d\mu}{dr} - \frac{1}{r^2 I} \right) \right]^{-1} = 0,$$

or as before

$$r^{-2} \left(\mu a \cos^2\phi + \frac{\mu \cos\phi}{r} + \sin^2\phi \cos\phi \frac{d\mu}{dr} \right)^{-1} + \int^r \frac{dr}{\mu r^2 \cos^2\phi}$$

is constant.

The other equation is in this case

$$\left\{ b' + \frac{\sin(\phi' + \delta)}{r' \sin\delta} \right\} \left\{ b - \frac{\sin(\phi - \delta)}{r \sin\delta} \right\} + \frac{\sin\phi \sin\phi'}{r r' \sin^2\delta} = 0,$$

which expresses that the secondary focus is always at the point of intersection of the tangent to the axial ray with the diameter through the initial secondary focus.

EVALUATION OF TWO DEFINITE INTEGRALS.

By A. R. Forsyth.

THE integrals in question are

$$\int_0^\pi \sin^m\theta e^{a\theta} d\theta, \quad \int_0^\pi \sin^m\theta \cos^n\theta e^{a\theta} d\theta,$$

where the indices m and n are not restricted to be integers.* They prove to be examples in which Briot and Bouquet's

* My attention was directed to the first of the two integrals considered, by a question from Prof. Karl Pearson.

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method of associating definite integrals with integrals of functions of complex variables can be used; an account of this method is given in their *Treatise on Elliptic Functions* (chapters I. and II. of Book III), and in my *Theory of Functions* (§ 25).

1. Let I denote the integral

$$\int_0^\pi \sin^m\theta e^{a\theta}d\theta,$$

where m is a quantity having its real part greater than -1, so that the modulus of I is finite in value. We have

$$2^m i^m I = \int_0^\pi (e^{\theta i} - e^{-\theta i})^m e^{a\theta}d\theta$$

$$= \int_0^\pi (e^{2\theta i} - 1)^m e^{(a-mi)\theta}d\theta.$$

Let $e^{\theta i} = z$, so that as θ increases from 0 to π, the variable z describes a semi-circle, centre the origin and radius unity; then

$$2^m i^{m+1} I = \int_1^{-1} (z^2 - 1)^m z^{-ai-m-1}dz,$$

with the specified path of integration.

The critical points of the subject of integration are $z = 0, \infty, \pm 1$. If we take a curve (fig. 41) composed of two small quadrants at 1 and -1, of two radii from 0 to 1 and from 0 to -1 respectively, of a small semi-circle centre the origin, and of a semi-circle centre the origin and radius unity, any branch of the subject of integration is uniform, finite and continuous within the enclosed area. Hence, by a well-known theorem† due to Cauchy, the integral round the boundary of the curve is zero.

Because the real part of m is greater than -1, the integral along the small quadrant at the point 1 is zero* in the limit when the radius of that quadrant is indefinitely diminished. Similarly the integral along the small quadrant at the point -1 is zero in the same limit. Likewise if the real part of $-ai-m$ is positive (so that if, for instance, a is real, we should take $0 > m > -1$), the integral round the small semi-circle quadrant at the origin is zero in the limit when the radius of that quadrant is indefinitely diminished.

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* See my *Theory of Functions*, p. 23. † *Ib.*, p. 85.

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Hence, denoting

$$(z^2-1)^m z^{-ai-m-1}$$

by $f(z)$, we have

$$\int_1^{-1} f(z)\,dz + \int_{-1}^{1} f(x)\,dx = 0,$$

so that

$$2^m i^{m+1} I = -\int_{-1}^{1} f(x)\,dx,$$

where in the latter integral the variables are real and the principal value is to be taken.

Let $$X = \int_0^1 f(x)\,dx,$$

so that

$$X = (-1)^m \int_0^1 (1-x^2)^m x^{-ai-m-1} dx$$

$$= \tfrac{1}{2} e^{m\pi i} \int_0^1 (1-u)^m u^{-\frac{1}{2}(ai+m)-1} du,$$

where $x^2 = u$. But

$$\int_0^1 u^{\mu-1}(1-u)^{\nu-1}\,du = B(\mu, \nu)$$

$$= \frac{\Pi(\mu-1)\,\Pi(\nu-1)}{\Pi(\mu+\nu-1)},$$

and therefore

$$X = \tfrac{1}{2} e^{m\pi i} \frac{\Pi(-\frac{1}{2}ai - \frac{1}{2}m - 1)\,\Pi(m)}{\Pi(\frac{1}{2}m - \frac{1}{2}ai)}.$$

Moreover,

$$\Pi(\sigma-1)\,\Pi(-\sigma) = \frac{\pi}{\sin\sigma\pi},$$

so that

$$\Pi(-\tfrac{1}{2}ai - \tfrac{1}{2}m - 1)\,\Pi(\tfrac{1}{2}ai + \tfrac{1}{2}m) = -\frac{\pi}{\sin\frac{1}{2}\pi(m+ai)},$$

whence

$$X = -\frac{\frac{1}{2}\pi e^{m\pi i}}{\sin\frac{1}{2}\pi(m+ai)} \frac{\Pi(m)}{\Pi(\frac{1}{2}m + \frac{1}{2}ai)\,\Pi(\frac{1}{2}m - \frac{1}{2}ai)}.$$

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Also

$$\int_{-1}^{0} f(x)\,dx = (-1)^m \int_{-1}^{0} (1-x^2)^m x^{-ai-m-1}\,dx$$

$$= (-1)^{-ai} \int_{1}^{0} (1-x^2)^m x^{-ai-m-1}\,dx,$$

where x is now positive throughout the range of integration, as in X; and therefore

$$\int_{-1}^{0} f(x)\,dx = -(-1)^{-ai} \int_{0}^{1} (1-x^2)^m x^{-ai-m-1}\,dx$$

$$= -\tfrac{1}{2} e^{\pi a} \int_{0}^{1} (1-u)^m u^{-\frac{1}{2}(ai+m)-1} du$$

$$= -\tfrac{1}{2} e^{\pi a} \frac{\Pi(-\frac{1}{2}ai - \frac{1}{2}m - 1)\,\Pi(m)}{\Pi(\frac{1}{2}m - \frac{1}{2}ai)}$$

$$= \frac{\frac{1}{2}\pi e^{\pi a}}{\sin\frac{1}{2}\pi(m+ai)} \frac{\Pi(m)}{\Pi(\frac{1}{2}m + \frac{1}{2}ai)\,\Pi(\frac{1}{2}m - \frac{1}{2}ai)}$$

as before. Thus

$$\int_{-1}^{1} f(x)\,dx = \left\{\int_{0}^{1} + \int_{-1}^{0}\right\} f(x)\,dx$$

$$= \frac{\frac{1}{2}\pi(e^{\pi a} - e^{m\pi i})}{\sin\frac{1}{2}\pi(m+ai)} \frac{\Pi(m)}{\Pi(\frac{1}{2}m + \frac{1}{2}ai)\,\Pi(\frac{1}{2}m - \frac{1}{2}ai)}$$

$$= -\pi i e^{\frac{1}{2}\pi m i + \frac{1}{2}\pi a} \frac{\Pi(m)}{\Pi(\frac{1}{2}m + \frac{1}{2}ai)\,\Pi(\frac{1}{2}m - \frac{1}{2}ai)}.$$

This is the value of

$$-2^m i^{m+1} I,$$

that is, of

$$-2^m i e^{\frac{1}{2}m\pi i} I;$$

and therefore

$$I = 2^{-m} \pi e^{\frac{1}{2}\pi a} \frac{\Pi(m)}{\Pi(\frac{1}{2}m + \frac{1}{2}ai)\,\Pi(\frac{1}{2}m - \frac{1}{2}ai)}.$$

The limitations under which this result has been established are that the real part of m lies between 0 and -1, and that the real part of $-ai-m$ is greater than 0.

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2. Now, if generally

$$I_m(a) = \int_0^\pi \sin^m\theta e^{a\theta} d\theta,$$

then

$$I_m(-a) = \int_0^\pi \sin^m\theta e^{-a\theta} d\theta$$

$$= e^{-a\pi} \int_0^\pi \sin^m\phi e^{a\phi} d\phi$$

on writing $\theta = \pi - \phi$, so that

$$I_m(-a) = e^{-a\pi} I_m(a),$$

and therefore

$$e^{\frac{1}{2}a\pi} I_m(-a) = e^{-\frac{1}{2}a\pi} I_m(a).$$

Thus, when a is real, $e^{-\frac{1}{2}a\pi} I_m(a)$ is an even function of a; and when a is a pure imaginary, say ia, then $e^{-\frac{1}{2}a\pi} I_m(a)$ is an even function of a. Also, the formula of reduction

$$I_m(a) = \frac{m(m-1)}{m^2 + a^2} I_{m-2}(a)$$

can be obtained, the only limitation being that the real part of $m-2$ is greater than -1. The preceding expression obtained for I satisfies this relation of reduction, so that the limitation that the real part of m must lie between 0 and -1 is not necessary for the validity of the result.

The simplest cases are those in which m is purely real—it must be greater than -1 in order that the integral may be finite. When a is zero, the integral becomes

$$\int_0^\pi \sin^m\theta d\theta,$$

the value of which is

$$= \pi^{\frac{1}{2}} \frac{\Pi(\frac{1}{2}m - \frac{1}{2})}{\Pi(\frac{1}{2}m)}$$

$$= \pi^{\frac{1}{2}} \frac{\Pi(\frac{1}{2}m - \frac{1}{2})\,\Pi(\frac{1}{2}m)}{\Pi(\frac{1}{2}m)\,\Pi(\frac{1}{2}m)}$$

$$= \frac{\pi}{2^m} \frac{\Pi(m)}{\Pi(\frac{1}{2}m)\,\Pi(\frac{1}{2}m)},$$

so that the result holds when a vanishes for any value of m that is greater than -1.

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Again

$$\Pi(\tfrac{1}{2}ai)\,\Pi(-\tfrac{1}{2}ai) = \tfrac{1}{2}ai\Pi(\tfrac{1}{2}ai-1)\,\Pi(-\tfrac{1}{2}ai)$$

$$= \frac{\pi ai}{2\sin(\frac{1}{2}a\pi i)}$$

$$= \frac{\pi a e^{\frac{1}{2}\pi a}}{e^{\pi a}-1},$$

and therefore

$$\frac{\pi e^{\frac{1}{2}\pi a}}{\Pi(\frac{1}{2}ai)\,\Pi(-\frac{1}{2}ai)} = \frac{1}{a}\,(e^{\pi a}-1)$$

which is the value of $\int_0^\pi e^{a\theta}d\theta$. The result is consequently true when m is zero, whatever a may be.

We therefore infer that *whatever be the value of* a, *the value of the integral*

$$\int_0^\pi \sin^m\theta e^{a\theta}d\theta$$

is

$$\frac{\pi e^{\frac{1}{2}\pi a}\Pi(m)}{2^m\Pi(\frac{1}{2}m+\frac{1}{2}ai)\,\Pi(\frac{1}{2}m-\frac{1}{2}ai)},$$

provided the real part of m *is greater than* -1.

3. Other integrals can be deduced by forming appropriate combinations; thus

$$\int_0^\pi \sin^m\theta\cos(\gamma\theta+\alpha)\,d\theta = \frac{\pi\Pi(m)\cos(\frac{1}{2}\pi\gamma+\alpha)}{2^m\Pi(\frac{1}{2}m+\frac{1}{2}\gamma)\,\Pi(\frac{1}{2}m-\frac{1}{2}\gamma)},$$

and

$$\int_0^\pi \sin^m\theta\cosh(\gamma\theta+\alpha)\,d\theta = \frac{\pi\Pi(m)\cosh(\frac{1}{2}\pi\gamma+\alpha)}{2^m\Pi(\frac{1}{2}m+\frac{1}{2}\gamma i)\,\Pi(\frac{1}{2}m-\frac{1}{2}\gamma i)},$$

where α and γ are constants and in each case the real part of the constant index m is greater than -1.

Similarly, by taking $\phi = \theta - \frac{1}{2}\pi$, we have

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^m\phi e^{a\phi}d\phi = \frac{\pi\Pi(m)}{2^m\Pi(\frac{1}{2}m+\frac{1}{2}ai)\,\Pi(\frac{1}{2}m-\frac{1}{2}ai)}$$

with the same single restriction on the value of m.

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Integrals of these types are given* by Bierens de Haan with the limitation that m is an integer.

4. Let G denote the integral

$$\int_0^\pi \sin^m\theta \cos^n\theta e^{a\theta i} d\theta,$$

where m and n are quantities having their real parts greater than -1, so that the modulus of G is finite in value. Making the same transformation as before, we find

$$2^{m+n} i^{m+1} G = \int_1^{-1} (z^2-1)^m (z^2+1)^n z^{-ai-m-n-1} dz,$$

where $e^{\theta i} = z$, so that the variable z describes a semicircle centre the origin and radius unity passing from 1 to -1.

The critical points of the subject of integration are $z = 0, \infty, \pm 1, \pm i$.

Take a curve (fig. 42) composed of two small quadrants at 1 and -1; two small semi-circles at i and 0; two quadrants, centre the origin and radius unity, from 1 to i and from i to -1 respectively; and two radii from 0 to 1 and from 0 to -1 respectively. Any branch of the subject of integration is uniform, finite and continuous within the enclosed area; the integral round the boundary of the curve vanishes, by Cauchy's Theorem.

As in the former case, the parts contributed to the integral by the small quadrants at 1 and -1 each vanish in the limit when the radius is made infinitesimal, because the real part of m is greater than -1; the part contributed by the small semi-circle at i vanishes in the limit when the radius is made infinitesimal, because the real part of n is greater than -1; and the part contributed by the small semi-circle at the origin vanishes in the limit when the radius is made infinitesimal, if the real part of $-ai-m-n$ be a positive quantity greater than zero. Assuming this condition satisfied—so that for instance if a be real we should have the limitations

$$m > -1, \; n > -1, \; 0 > m+n,$$

so far as the real parts of m and n are concerned, we have

$$\int_1^{-1} (z^2-1)^m (z^2+1)^n z^{-ai-m-n-1} dz + \int_{-1}^{1} (x^2-1)^m (x^2+1)^n x^{-ai-m-n-1} dx = 0,$$

* *Exposé de la Theorie des Intégrales Définies*, pp. 244, 251.

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and therefore

$$2^{m+n}i^{m+1}G = -\int_{-1}^{1}(x^2-1)^m(x^2+1)^n x^{-\alpha i-m-n-1}dx.$$

The range of integration on the right-hand side being taken from -1 to 0 and from 0 to 1, we have

$$\int_0^1 (x^2-1)^m (x^2+1)^n x^{-\alpha i-m-n-1}dx$$

$$= (-1)^m \int_0^1 f(x)\,dx,$$

where $f(x)$ denotes

$$(1-x^2)^m (1+x^2)^n x^{-\alpha i-m-n-1}$$

for real values of x lying between 0 and 1. Also

$$\int_{-1}^{0}(x^2-1)^m(x^2+1)^n x^{-\alpha i-m-n-1}dx$$

$$= (-1)^m \int_{-1}^{0}(1-x^2)^m(1+x^2)^n x^{-\alpha i-m-n-1}dx$$

$$= -(-1)^{-\alpha i-n}\int_0^1 f(x)\,dx.$$

Hence

$$2^{m+n}i^{m+1}G = \{(-1)^{-\alpha i-n}-(-1)^m\}\int_0^1 f(x)\,dx$$

$$= (e^{\alpha\pi-n\pi i}-e^{m\pi i})\int_0^1 f(x)\,dx.$$

Now

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}\int_0^1 (1-u)^m(1+u)^n u^{-\frac{1}{2}(\alpha i+m+n)-1}du$$

$$= 2^{n-1}\int_0^1 v^m(1-v)^{-\frac{1}{2}(\alpha i+m+n)-1}(1-\tfrac{1}{2}v)^n dv,$$

where $x^2 = u = 1-v$. But it is known that

$$\int_0^1 v^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xv)^{-\alpha}dv$$

$$= \frac{\Pi(\gamma-1)}{\Pi(\beta-1)\,\Pi(\gamma-\beta-1)}F(\alpha,\beta,\gamma,x)$$

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when the real parts of β and of $\gamma - \beta$ are positive and F denotes the hypergeometric series of argument x, the modulus of x being supposed not greater than unity. The conditions are satisfied for the preceding integral; so that denoting

$$F(-n,\ 1+m,\ 1+\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2}ai,\ \tfrac{1}{2})$$

by $F_{\frac{1}{2}}$, we have

$$\int_0^1 f(x)\,dx = 2^{n-1}\,\frac{\Pi(m)\,\Pi(-\frac{1}{2}ai-\frac{1}{2}m-\frac{1}{2}n-1)}{\Pi(-\frac{1}{2}ai+\frac{1}{2}m-\frac{1}{2}n)}\,F_{\frac{1}{2}}.$$

Also

$$\Pi(-\tfrac{1}{2}ai-\tfrac{1}{2}m-\tfrac{1}{2}n-1)\,\Pi(\tfrac{1}{2}ai+\tfrac{1}{2}m+\tfrac{1}{2}n)$$

$$=\frac{\pi}{-\sin\frac{1}{2}\pi(ai+m+n)}$$

$$=\frac{2i\pi}{e^{\pi a-n\pi i}-e^{m\pi i}}\,e^{\frac{1}{2}\pi a+\frac{1}{2}m\pi i-\frac{1}{2}n\pi i},$$

so that

$$\int_0^1 f(x)\,dx = 2^n i\pi\,\frac{e^{\frac{1}{2}\pi a+\frac{1}{2}m\pi i-\frac{1}{2}n\pi i}}{e^{\pi a-n\pi i}-e^{m\pi i}}\,\frac{\Pi(m)\,F_{\frac{1}{2}}}{\Pi(\frac{1}{2}m-\frac{1}{2}ai-\frac{1}{2}n)\,\Pi(\frac{1}{2}m+\frac{1}{2}ai+\frac{1}{2}n)}.$$

Thus finally

$$G = e^{\frac{1}{2}\pi(a-n)i}\,\frac{2^{-m}\pi\,\Pi(m)}{\Pi(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}ai)\,\Pi(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}ai)\,F_{\frac{1}{2}}}.$$

The result has been established under certain limitations as to m, n, a; but considerations, similar to those adduced before, lead to the inference that the only limitations necessary to ensure that *the expression on the right-hand side is the value of*

$$\int_0^\pi \sin^m\theta\,\cos^n\theta e^{a\theta}d\theta$$

are that the real parts of the quantities m and n are each greater than -1.

5. As a special case of some interest, let $a=0$; the integral becomes

$$\int_0^\pi \sin^m\theta\,\cos^n\theta d\theta$$

$$=\int_0^{\frac{1}{2}\pi}\sin^m\theta\,\cos^n\theta d\theta+(-1)^n\int_0^{\frac{1}{2}\pi}\sin^n\theta\,\cos^m\theta d\theta$$

$$=\tfrac{1}{2}(1+e^{-n\pi i})\,\frac{\Pi(\frac{1}{2}n-\frac{1}{2})\,\Pi(\frac{1}{2}m-\frac{1}{2})}{\Pi(\frac{1}{2}m+\frac{1}{2}n)}.$$

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Equating the results for the present form, we have

$$F(-n, 1+m, 1+\tfrac{1}{2}m-\tfrac{1}{2}n, \tfrac{1}{2})$$

$$=\frac{2^m \cos\tfrac{1}{2}n\pi}{\pi}\,\frac{\Pi\left(\frac{n-1}{2}\right)\Pi\left(\frac{m-1}{2}\right)\Pi\left(\frac{m-n}{2}\right)}{\pi(m)},$$

which after some reduction leads to

$$F(-n, 1+m, 1+\tfrac{1}{2}m-\tfrac{1}{2}n, \tfrac{1}{2})=\frac{\pi^{\frac{1}{2}}\Pi\left(\frac{m-n}{2}\right)}{\Pi\left(-\frac{n+1}{2}\right)\Pi\left(\frac{m}{2}\right)}.$$

Taking $-n=2\alpha$, $1+m=2\beta$, the equation is transformed to

$$F(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}, \tfrac{1}{2})=\frac{\Pi(\alpha+\beta-\frac{1}{2})}{\Pi(\alpha-\frac{1}{2})\Pi(\beta-\frac{1}{2})}\,\pi^{\frac{1}{2}},$$

a result established by Gauss.*

Owing the generality of the main result, a number of special cases can easily be deduced.

NOTE TO MY PROOF OF EUCLID'S TWELFTH AXIOM.

By Thos. Cullovin.

Proposition. *If infinity in a given straight line be defined as a single point beyond which there is no other, two straight lines cannot meet at infinity otherwise than by never meeting.*

Let CD (fig. 41) be the nearest possible parallel to AB through the point C; and let CE be, if possible, a straight line passing through the same point, meeting AB at ∞ but

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* Ges. *Werke*, t. III., p. 227.

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not coinciding with CD. Then, since CE and CD do not coincide, they must make an angle with one another. Let CF be the line bisecting that angle. Then, since CD is the nearest line to AB drawn through C which never meets it, CF must meet AB. But since CF cannot meet CE at a finite distance, it must meet CE and AB at ∞. Apply CD to CE so that the angles ECF and DCF may coincide and it will follow that CD will meet CF at ∞. But CF meets CE at ∞ and CE meets AB at ∞. Therefore CD meets AB at ∞; but (hyp.) they never meet. Therefore, either the conception of straight lines meeting at ∞ must be rejected as implying an impossibility, or parallels must be regarded as meeting at ∞.

I conclude from the foregoing that it is inadmissible to treat the meeting of straight lines at ∞ as differing in any way from their never meeting. When infinity is defined to be a single point and parallels and intersecting lines are assumed to meet at ∞, such assumptions constitute a system of geometry differing radically from Euclid's. All Euclidean demonstrations involving the terms differently defined, would require to be remodelled in the new system. In the Greek geometry we cannot conceive straight lines as meeting otherwise than at finite distances from given points. This enables me to answer in advance the only objection against my proof of the theory of parallels that I am able to anticipate.

Objection. You have narrowed down the difficulty to one critical point in the bisecting line, and you have shown that a last possible line meeting the containing lines ϕ can not pass through that point; but it is equally reasonable to suppose that the first possible line that fails may pass through it. How do you show that such may not be the case, and that the lines meeting ϕ may not approach the point nearer than any given finite distance without ever reaching it?

Answer. I have shown that all conceivable perpendiculars to the bisector belong to one or other of two distinct classes, that the lines which meet cannot intermingle with those supposed to fail, and that no portion or point of the bisector can remain neutral. Consequently, the last line meeting would coincide with the first line failing, as no portion of the bisector could possibly intervene. In other words, there can be no first line failing.

To make this plainer let $P'Q'$ be a perpendicular (fig. 42) meeting, and PQ one failing, both at some distance from the

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critical point. Let PQ, of unlimited length, move along the bisector till it reaches the position $P'Q'$. At some of its point progress it must meet the containing lines for the first time as $P''Q''$. From the proposition I have last proved, the distances RS, BS, BR and BT must be regarded as finite, and the point R corresponds with the critical point P of my demonstration.

NOTE ON THE SMALL OSCILLATIONS OF THE FIRST ORDER OF KIRCHOFF'S ELLIPTIC VORTEX CYLINDER.

By P. H. Cowell, B.A., Scholar of Trinity College, Cambridge.

THE steady motion of the vortex is given by the velocity potentials

$$\psi' = -\frac{\zeta}{a+b}(bx^2 + ay^2) \text{ inside,}$$

and

$$\psi = -\zeta ab.\xi - \tfrac{1}{2}\zeta ab.e^{-2\xi}\cos 2\eta \text{ outside,}$$

where a, b are the semi-axes of the cross section, x, y the coordinates of any point referred to these axes, ξ, η conjugate functions given by the equation

$$x + iy = c\cosh(\xi + i\eta),$$

where

$$c^2 = a^2 - b^2,$$

and ζ the molecular rotation, and $\xi = \xi_0$ will be taken to denote the surface of separation in the steady motion.

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The motion is steady, and the ellipse retaining an invariable form rotates with angular velocity

$$\omega = \frac{2ab}{(a+b)^2}\zeta.$$

Mr. Love has shown that* there can exist small oscillations given by the velocity potentials $\psi' + \delta\psi'$ inside and $\psi + \delta\psi$ outside when,

$$\delta\psi' = \Sigma_m [A_m \cosh m\xi \cos m\eta + B_m \sinh m\xi \sin m\eta],$$

$$\delta\psi = \Sigma_m [A_m \cosh m\xi_0 . e^{-m(\xi-\xi_0)} \cos m\eta + B_m \sinh m\xi_0 e^{-m(\xi-\xi_0)} \sin m\eta].$$

In the most general small oscillation, the sign of summation includes all integral positive values of m. The normal types are found by giving m one single positive integral value only.

Mr. Love obtains the equations

$$\frac{dA_m}{dt} + \left[\frac{2\zeta . mab}{(a+b)^2} - 2\zeta . e^{-m\xi_0} \sinh m\xi_0\right] B_m = 0,$$

$$\frac{dB_m}{dt} - \left[\frac{2\zeta . mab}{(a+b)^2} - 2\zeta . e^{-m\xi_0} \cosh m\xi_0\right] A_m = 0.$$

From these equations he shows that, except for very flat ellipses whose axes are in a ratio greater than 3 : 1, the oscillations of the third and higher orders are stable. An initial displacement of the second order either indicates a rotational displacement of the ellipse as a whole, which does not affect the small oscillations, or else a deformation into another ellipse of slightly different ellipticity, which has a steady motion of its own, slightly different from that of the original ellipse.

We here consider the oscillations of the first order. Mr. Love has shown that they are stable for all eccentricities, and that their periodic time is that of a complete revolution of the elliptic figure of the vortex.

In this case, we have

$$\delta\psi_1' = A_1 \cosh\xi \cos\eta + B_1 \sinh\xi \sin\eta$$

$$= \frac{A_1 x + B_1 y}{c},$$

* *Proc. Lond. Math. Soc.*, Vol. xxv. (1893), p. 24.

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denoting a translational motion of the ellipse as a whole with velocity B_1/c parallel to the major axis, $-A_1/c$ parallel to the minor axis.

We also have

$$\frac{dA_1}{dt} - \frac{2\zeta b^2}{(a+b)^2} B_1 = 0,$$

$$\frac{dB_1}{dt} + \frac{2\zeta a^2}{(a+b)^2} A_1 = 0;$$

whence

$$A_1 = -Cca \cos(\omega t + \varepsilon),$$

$$B_1 = Ccb \sin(\omega + \varepsilon),$$

C and ε being arbitrary constants.

Hence, the ellipse, as a whole, moves with velocities $Cb \sin(\omega t + \varepsilon)$, $Ca \cos(\omega t + \varepsilon)$ parallel to the axes.

The components parallel to the axes at time $t = 0$, are

$$C[b \sin(\omega + \varepsilon) \cos \omega t - a \cos(\omega t + \varepsilon) \sin \omega t],$$

and

$$C[b \sin(\omega t + \varepsilon) \sin \omega t + a \cos(\omega t + \varepsilon) \cos \omega t].$$

These expressions denote an oscillatory motion superimposed on a steady motion, whose components are

$$\tfrac{1}{2} C[b \sin \varepsilon + a \cos \varepsilon],$$

$$\tfrac{1}{2} C[b \cos \varepsilon - a \sin \varepsilon].$$

By properly choosing the origin of time, we can make ε what we please.

Putting therefore

$$Cb \cos \varepsilon = Ca \sin \varepsilon = 2D.ab,$$

the non-periodic part of the motion is $D(a^2 + b^2)$ parallel to the x axis, and the periodic terms become

$$-D(a^2 + b^2)^{\frac{1}{2}} (a - b) \sin(2\omega + \varepsilon)$$

$$+D(a^2 + b^2)^{\frac{1}{2}} (a + b) \cos(2\omega t + \varepsilon).$$

Since the non-periodic part of the motion is always greater than the component in the reversed direction of the periodic part, the centre of the ellipse describes a wavy line without loops.

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NOTE ON THE ARITHMETICAL THEORY OF CONJUGATE BINARY QUADRATIC FORMS.

By G. B. Mathews, M.A.

IN Art. 123 of Smith's *Report on the Theory of Numbers*, (Coll. Papers i. p. 284) will be found the following proposition:—

"If (a, b, c), (a', b', c') are two primitive forms of the determinants D and D', whose joint invariant $ac' - 2bb' + ca'$ is zero, and if m and m' are the greatest common divisors of $a, 2b, c$; $a', 2b', c'$; then m^2D' and m'^2D are capable of primitive representation by the duplicates of (a, b, c) and (a', b', c') respectively."

It is the object of the present note to supply an elementary proof of this theorem. It will be convenient to say that two forms are *conjugate* when their joint invariant vanishes; and also to employ the symbol $dv(p, q, r, \ldots)$ to denote the greatest common divisor of p, q, r, etc.

If now we write

$$\left.\begin{aligned} f &= (a, b, c) \\ \sigma &= dv(a, c) \\ m &= dv(a, 2b, c) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

it will be possible to find integers h, k prime to each other, and such that

$$m(hc - ka) = 2\sigma b \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and if we put

$$f_1 = (mh, \sigma, -mk),$$

this is a form which is primitive and conjugate to f.

Another form of the same kind is

$$f_1 = \left(\frac{a}{\sigma}, 0, -\frac{c}{\sigma}\right).$$

By a known algebraical theorem, the most general form f' which is conjugate to f, is

$$\begin{aligned} f' &= tf_1 + uf_2 \\ &= \left(\frac{at}{\sigma} + mhu, \sigma u, -\frac{ct}{\sigma} - mku\right), \end{aligned}$$

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t, u being any quantities whatever. For the arithmetical application the coefficients of f' must be integral, hence we infer that

$$\sigma u,\ \sigma t/m,\ 2bt/m$$

are integers; since σ/m is prime to $2b/m$, t must be an integer. Finally, since mhu, mku are integers and h, k are prime to each other, mu is an integer.

Thus, with a change of notation, we may write

$$f' = \left(\frac{a\lambda}{\sigma} + h\mu,\ \frac{\sigma\mu}{m},\ -\frac{c\lambda}{\sigma} - k\mu\right) \ldots\ldots\ldots\ldots(3),$$

λ, μ being any integers; and this is the most general expression for a form conjugate to f.

If D' is the determinant of f', we have identically

$$m^2D' = \left(\frac{ac}{\sigma^2},\ \frac{m(hc+ka)}{2\sigma},\ \sigma^2 + m^2hk\right)(m\lambda,\ \mu)^2 \ldots(4).$$

which, with $m\lambda$, μ for the variables, is a representation of m^2D' by the form

$$\phi = \left(\frac{ac}{\sigma^2},\ \frac{m(hc+ka)}{2\sigma},\ \sigma^2 + m^2hk\right)\ldots\ldots\ldots(5).$$

It is easily seen with the help of (2) that the determinant of ϕ is D, and that its coefficients are all integers, the middle one being expressible as

$$B = b + \frac{mka}{\sigma} = -b + \frac{mhc}{\sigma} \ldots\ldots\ldots\ldots\ldots(6).$$

Moreover, if we compound the forms

$$(a,\ b,\ c),\ (c,\ -b,\ a)$$

by the process explained in Art. 110 of the Report, the resulting form is $(A,\ B,\ C)$, with

$$A = \frac{ac}{\sigma^2},$$

$$\left.\begin{aligned} \frac{aB}{\sigma} &\equiv -\frac{ab}{\sigma} \\ \frac{cB}{\sigma} &\equiv \frac{bc}{\sigma} \end{aligned}\right\} (\text{mod. } A),$$

$$C = \frac{B^2 - D}{A};$$

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and since the congruential conditions are in fact satisfied by the value of B given in (6), it follows that ϕ is compounded of (a, b, c) and $(c, -b, a)$; or, since the forms (a, b, c), $(c, -b, a)$ are properly equivalent, ϕ is equivalent to the duplicate of (a, b, c).

The representation of m^2D' is primitive so long as $m\lambda$ is prime to μ; and it may be shown from (3) that this is always the case when f' is primitive.

On account of the symmetrical form of the joint invariant we may conclude without further analysis that m'^2D is expressible by a form which is a duplicate of f'.

As a numerical example, which will illustrate some points of detail not expressly referred to in the above, let

$$f = (2, 3, 1), \quad f' = (12, 7, 15);$$

then the reduction of f' to the form (3) is effected by putting

$$\lambda = -1 - 7t, \qquad h = 2 + 2t,$$

$$\mu = 7, \qquad k = -2 + t,$$

where t is any integer; and then we have identically

$$-131 = (2, -1 + 2t, -3 - 2t + 2t^2)(-1 - 7t, 7)^2.$$

As a special case, $t = 0$ gives

$$-131 = 2.(-1)^2 - 2.(-1).7 - 3.7^2.$$

Again, if we put

$$f = (12, 7, 15), \quad f' = (2, 3, 1),$$

and proceed as before, we have in the same notation

$$\lambda = -t, \qquad h = 4t + 2,$$

$$\mu = 1, \qquad k = 5t - 1;$$

and thence $7 = (20, 3 + 20t, 7 + 6t + 20t^2)(-t, 1)^2$.

Two conjugate forms cannot both be definite; because, if

$$ac' + a'c - 2bb' = 0,$$

then $$4b^2b'^2 = (ac' - a'c)^2 + 4aca'c',$$

and therefore we cannot have simultaneously

$$b^2 < ac, \quad b'^2 < a'c'.$$

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The Jacobian of two forms is conjugate to each of them; when the forms are conjugate to each other, $ac - a'c$ is even, and if we take the Jacobian in the determinate form

$$j = \frac{\partial(f, f')}{\partial(x, y)} = \{ab' - a'b, \tfrac{1}{2}(ac' - a'c), bc' - b'c\},$$

its coefficients are integers, and its determinant is $-DD'$.

Thus j is definite or indefinite according as f, f' are both indefinite or one definite and the other indefinite. Two conjugate forms and their Jacobian may in a certain sense be regarded as a closed group; the Jacobian of each pair of forms coinciding, up to a numerical factor, with the remaining form.

If, as usual, we represent a definite form by the pair of points whose affixes are the roots of the form, and an indefinite form (a, b, c) by the circle

$$a(x^2 + y^2) + 2bx + c = 0,$$

the condition

$$ac' + a'c - 2bb' = 0$$

expresses, when f, f' are both indefinite, that their representative circles cut orthogonally; and, when one form is definite and the other indefinite, that the representative points of the one form lie on the representative circle of the other.

Thus two conjugate forms and their Jacobian are geometrically represented by two orthotomic circles and their points of intersection.

Let us suppose, as before, that the form $f = (a, b, c)$ is given, and that we have chosen a determinate pair of integers h, k satisfying the relation (2). Then any form f', conjugate to f, is determined by the integers λ, μ in the identity (3). For convenience, we will say that λ, μ are the *coordinates* of f.

If we again use j to denote the Jacobian, it is found by direct calculation that the coordinates of j are

$$\left.\begin{aligned} \lambda' &= B\lambda + \frac{C}{m}\mu \\ \mu' &= -mA\lambda - B\mu \end{aligned}\right\} \ldots\ldots\ldots\ldots \ldots(7),$$

where A, B, C are the coefficients of ϕ in the equation (5) above.

It may be observed that λ', μ' are derived from λ, μ by a

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linear transformation of which the determinant is $-D$, and also that

$$\lambda'\mu - \lambda\mu' = mD' \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

by equation (4).

By ascribing to λ, μ all possible integral values, we obtain an infinite number of forms all conjugate to f. Those forms f' for which D' has different values are entirely distinct from each other: but the question arises whether the forms f', which have the same determinant, are or are not equivalent.

To fix the ideas let us suppose that f is properly primitive and indefinite: then ϕ is also properly primitive and indefinite. Suppose that (λ_0, μ_0) is a special pair of integers giving a primitive representation of D' in the form

$$D' = (A, B, C)(\lambda_0, \mu_0)^2.$$

Then we have a corresponding form f_0', with determinant D' and coordinates (λ_0, μ_0), which is conjugate to f.

From the special representation of D' just written down, we can deduce the general representation by putting

$$\left.\begin{aligned} \lambda &= (t - uB)\lambda_0 - uC\mu_0 \\ \mu &= uA\lambda_0 + (t + uB)\mu_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (9),$$

where t, u are any integers such that

$$t^2 - Du^2 = 1.$$

If j_0 is the Jacobian of f and f_0', it follows from the preceding determination of the coordinates of j_0 that the form f' whose coordinates are λ and μ is

$$f' = tf_0' - uj_0 \quad \ldots\ldots\ldots\ldots\ldots (10).$$

We thus obtain a set of forms f' corresponding to a set of representations of D' by ϕ.

Two conjugate forms f, f' remain conjugate when they are subjected to the same linear transformation. When the transformation leaves f unaltered, we thus obtain from f' another form f'' equivalent to f' and conjugate to f.

Let us write $f' = (a', b', c')$ and apply to it the substitution

$$\begin{vmatrix} t - bu, & -cu \\ au & , t + bu \end{vmatrix} \quad [t^2 - Du^2 = 1],$$

which leaves f unaltered. Then actually working out the transformation, and taking account of the fact that f, f' are conjugate, we find that f'' becomes

$$f'' = (2t^2 - 1)f' + 2tuj.$$

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If (T_1, U_1) is the fundamental solution of $t^2 - Du^2 = 1$, the other solutions are obtained by putting

$$T_s + U_s\sqrt{D} = (T_1 + U_1\sqrt{D})^s \quad [s = 2, 3, 4\ldots],$$

and hence also if $t = T_s$, $u = U_s$,

$$2t^2 - 1 = T_{2s}, \; 2tu = U_{2s}.$$

Thus if we write

$$\begin{vmatrix} T_s - bU_s & -cU_s \\ aU_s & T_s + bU_s \end{vmatrix} = S_s,$$

we have the curious result that

$$S_s(f') = T_{2s}f' + U_{2s}j \ldots\ldots\ldots\ldots\ldots (11),$$

f, f' being conjugate, and

$$j = \frac{\partial(f, f')}{\partial(x, y)}.$$

As a numerical verification, suppose

$$f = (2, 3, 1), \; f' = (12, 7, 15);$$

then

$$j = (-22, 9, 38), \; D = 7;$$

$$T_1 = 8, \; U_1 = 3, \; T_2 = 127, \; U_2 = 48.$$

The substitution S_1 is

$$\begin{vmatrix} -1, & -3 \\ 6, & 17 \end{vmatrix}$$

and the result of applying this to f' is

$$(468, 1321, 3729),$$

which agrees, as it should do, with $127f' + 48j$.

From a comparison of (10) and (11) we infer that the forms f' derived from the different representations of D' by ϕ fall into two groups, the forms of each group being properly equivalent. The two groups are distinct, at least in general: thus, for instance, in the above numerical example,

$$T_1 f' + U_1 j = (30, 83, 234),$$

which is conjugate to f, but not properly equivalent to f'.

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ON THE SIXTY ICOSAHEDRAL SUBSTITUTIONS.

By Professor Cayley.

THE Sixty Icosahedral Substitutions were obtained in an elegant form by Gordan in the paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Math. Ann.*, t. XII. (1877), pp. 23–46, see p. 45, where the group is exhibited in the canonical form

$$\varepsilon^{\mu}\eta, \quad \frac{-\varepsilon^{\mu}}{\eta}, \quad \varepsilon^{\mu}\frac{(\varepsilon^{2}+\varepsilon^{4})\eta+\varepsilon^{\rho}}{\varepsilon^{-\rho}\eta-(\varepsilon+\varepsilon^{3})}, \quad \varepsilon^{\mu}\frac{-\varepsilon^{-\rho}\eta+(\varepsilon+\varepsilon^{3})}{(\varepsilon^{2}+\varepsilon^{4})\eta+\varepsilon^{\rho}}$$

(ε an imaginary fifth root of unity), agreeing with the developed form given in my paper, "On the Schwarzian Derivative and the Polyhedral Functions," *Camb. Phil. Trans.*, t. XIII., Pt. 1 (1881), pp. 5–68, see pp. 55–56.

But a slight change of form is desirable. I write $\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$ to denote the unimodular matrix

$$\begin{Bmatrix} \dfrac{a}{\sqrt{(ad-bc)}}, & \dfrac{b}{\sqrt{(ad-bc)}} \\ \dfrac{c}{\sqrt{(ad-bc)}}, & \dfrac{d}{\sqrt{(ad-bc)}} \end{Bmatrix},$$

so that the value is independent of the absolute magnitudes of a, b, c, d, but depends only on their ratios; and observe that $m\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$ denotes $\begin{Bmatrix} ma, & mb \\ c & d \end{Bmatrix}$, the factor m applying to the top-line only. As before, ε denotes an imaginary fifth rood of unity, and I write $A=\varepsilon+\varepsilon^{4}$, $B=\varepsilon^{2}+\varepsilon^{3}$, $A'=\varepsilon-\varepsilon^{4}$, $B'=\varepsilon^{2}-\varepsilon^{3}$, but, in fact, I scarcely need more than the first of these symbols.

The substitutions are

$$(x_1, y_1) = (a, \ b \mid c, \ d)(x, y),$$

that is, $x_1=ax+by$, $y_1=cx+dy$; and we then have sixty matrices $\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$ forming the group in question.

Giving to ρ the values 0, 1, 2, 3, 4, I write

$$K_{\rho} = \begin{Bmatrix} \varepsilon^{\rho}, & 0 \\ 0, & \varepsilon^{-\rho} \end{Bmatrix},$$

so that K_0 is the matrix unity $\begin{Bmatrix} 1, & 0 \\ 0, & 1 \end{Bmatrix}$,

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$$L_\rho = \begin{Bmatrix} 0, & -\varepsilon^\rho \\ \varepsilon^{-\rho}, & 0 \end{Bmatrix},$$

$$M_\rho = \begin{Bmatrix} A, & \varepsilon^\rho \\ \varepsilon^{-\rho}, & -A \end{Bmatrix}, \quad \text{mod.} \begin{vmatrix} A, & \varepsilon^\rho \\ \varepsilon^{-\rho}, & A \end{vmatrix} \text{ is } = -A^2 - 1, \ = B^2,$$

$$N_\rho = \begin{Bmatrix} -\varepsilon^\rho, & A \\ A, & \varepsilon^{-\rho} \end{Bmatrix}, \quad \text{mod.} \begin{vmatrix} -\varepsilon^\rho, & A \\ A, & \varepsilon^{-\rho} \end{vmatrix} \text{ is } \ -A^2 - 1, \ = B^2,$$

then the matrices are

$$\begin{array}{l} K_0,\ K_1,\ K_2,\ K_3,\ K_4 \\ L_0,\ L_1,\ L_2,\ L_3,\ L_4 \\ M_0,\ M_1,\ M_2,\ M_3,\ M_4 \end{array}$$

$$\begin{array}{c|c|l} 1 & & \text{read } M_0, M_1, \ldots, \\ \varepsilon & & \quad \varepsilon M_0, \\ \varepsilon^2 & & \quad \vdots \\ \varepsilon^3 & & \\ \varepsilon^4 & & \end{array}$$

$$\begin{array}{l} N_0,\ N_1,\ N_2,\ N_3,\ N_4 \end{array}$$

$$\begin{array}{c|c|l} 1 & & \text{,, } N_0, N_1, \ldots, \\ \varepsilon & & \quad \varepsilon N_0, \\ \varepsilon^2 & & \quad \vdots \\ \varepsilon^3 & & \\ \varepsilon^4 & & \end{array}$$

viz. $5 + 5 + 25 + 25$, 60 in all matrices, including the matrix unity.

It is to be observed that the system must remain unaltered if we change ε into ε^2, ε^3, or ε^4: this is at once obvious for the change ε into ε^4, but less immediately so for the other two changes. Suppose ε changed into ε^2, then M_ρ becomes

$$= \begin{Bmatrix} B, & \varepsilon^{2\rho} \\ \varepsilon^{-2\rho}, & -B \end{Bmatrix}, \ = \begin{Bmatrix} AB, & A\varepsilon^{2\rho} \\ A\varepsilon^{-2\rho}, & -AB \end{Bmatrix} = \begin{Bmatrix} 1, & A\varepsilon^{2\rho} \\ A\varepsilon^{-2\rho}, & -1 \end{Bmatrix} \text{ (since } AB =) 1$$

$$= \varepsilon^{4\rho} \begin{Bmatrix} -\varepsilon^{-2\rho}, & A \\ A, & \varepsilon^{2\rho} \end{Bmatrix}, \text{ or changing } \rho \text{ into } -\tfrac{1}{2}\rho, \text{ which has the}$$

same values, this is $= \varepsilon^{-2\rho} \begin{Bmatrix} -\varepsilon^\rho, & A \\ A, & \varepsilon^{-\rho} \end{Bmatrix}$, viz. M_ρ is equal to a power of ε into N_ρ: and similarly N_ρ becomes equal to a power of ε into M_ρ. And the like as regards the change ε into ε^3.

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I find that the sixty matrices may be coordinated with the sixty positive substitutions of the letters *abcde* as follows:

1	$K_0 \begin{Bmatrix} 1, & 0 \\ 0, & 1 \end{Bmatrix}$	1			
2	$L_0 \begin{Bmatrix} 0, & -1 \\ 1, & 0 \end{Bmatrix}$	$ad.bc$	17	$\varepsilon^3 M_0, \varepsilon^3 \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	ced
3	$L_1 \begin{Bmatrix} 0, & -\varepsilon \\ \varepsilon^4, & 0 \end{Bmatrix}$	$ab.de$	18	$\varepsilon^2 M_1, \varepsilon^2 \begin{Bmatrix} A, & \varepsilon \\ \varepsilon^4-A & \end{Bmatrix}$	adc
4	$L_2 \begin{Bmatrix} 0, & -\varepsilon^2 \\ \varepsilon^3, & 0 \end{Bmatrix}$	$ac.be$	19	$\varepsilon^2 M_2, \varepsilon^2 \begin{Bmatrix} A, & \varepsilon^2 \\ \varepsilon^3-A & \end{Bmatrix}$	adb
5	$L_3 \begin{Bmatrix} 0, & -\varepsilon^3 \\ \varepsilon^2, & 0 \end{Bmatrix}$	$bd.ce$	20	$\varepsilon^2 M_3, \varepsilon^2 \begin{Bmatrix} A, & \varepsilon^4 \\ \varepsilon^3-A & \end{Bmatrix}$	acb
6	$L_4 \begin{Bmatrix} 0, & -\varepsilon^4 \\ \varepsilon, & 0 \end{Bmatrix}$	$ae.cd$	21	$\varepsilon^2 M_4, \varepsilon^2 \begin{Bmatrix} A, & \varepsilon^4 \\ \varepsilon-A & \end{Bmatrix}$	bce
7	$M_0 \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	$ac.bd$	22	$\varepsilon^3 M_0, \varepsilon^3 \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	abc
8	$M_1 \begin{Bmatrix} A, & \varepsilon \\ \varepsilon^4-A & \end{Bmatrix}$	$ae.bd$	23	$\varepsilon^3 M_1, \varepsilon^3 \begin{Bmatrix} A, & \varepsilon \\ \varepsilon^4-A & \end{Bmatrix}$	bec
9	$M_2 \begin{Bmatrix} A, & \varepsilon^2 \\ \varepsilon^3-A & \end{Bmatrix}$	$ae.bc$	24	$\varepsilon^3 M_2, \varepsilon^3 \begin{Bmatrix} A, & \varepsilon^2 \\ \varepsilon^3-A & \end{Bmatrix}$	cde
10	$M_3 \begin{Bmatrix} A, & \varepsilon^3 \\ \varepsilon^2-A & \end{Bmatrix}$	$bc.de$	25	$\varepsilon^3 M_3, \varepsilon^3 \begin{Bmatrix} A, & \varepsilon^3 \\ \varepsilon^2-A & \end{Bmatrix}$	acd
11	$M_4 \begin{Bmatrix} A, & \varepsilon^4 \\ \varepsilon-A & \end{Bmatrix}$	$ac.de$	26	$\varepsilon^3 M_4, \varepsilon^3 \begin{Bmatrix} A, & \varepsilon^4 \\ \varepsilon-A & \end{Bmatrix}$	abd
12	$N_0 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	$ab.cd$	27	$\varepsilon N_0, \varepsilon \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	aec
13	$N_1 \begin{Bmatrix} -1, & A\varepsilon \\ A\varepsilon^4, & 1 \end{Bmatrix}$	$ad.be$	28	$\varepsilon^4 N_1, \varepsilon^4 \begin{Bmatrix} -\varepsilon, & A \\ A, & \varepsilon^4 \end{Bmatrix}$	bcd
14	$N_2 \begin{Bmatrix} -1, & A\varepsilon^2 \\ A\varepsilon^3, & 1 \end{Bmatrix}$	$ab.ce$	29	$\varepsilon^2 N_2, \varepsilon^2 \begin{Bmatrix} -\varepsilon^2, & A \\ A, & \varepsilon^3 \end{Bmatrix}$	abc
15	$N_3 \begin{Bmatrix} -1, & A\varepsilon^3 \\ A\varepsilon^2, & 1 \end{Bmatrix}$	$be.cd$	30	$N_3, \begin{Bmatrix} -\varepsilon^3, & A \\ A, & \varepsilon^2 \end{Bmatrix}$	aed
16	$N_4 \begin{Bmatrix} -1, & A\varepsilon^4 \\ A\varepsilon, & 1 \end{Bmatrix}$	$ad.ce$	31	$\varepsilon^3 N_4, \varepsilon^3 \begin{Bmatrix} -\varepsilon^4, & A \\ A, & \varepsilon \end{Bmatrix}$	bed

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32	$\varepsilon^4 N_0,\ \varepsilon^4 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	*bde*
33	$\varepsilon^2 N_1,\ \varepsilon^2 \begin{Bmatrix} -\varepsilon, & A \\ A, & \varepsilon^4 \end{Bmatrix}$	*acb*
34	$N_2,\ \begin{Bmatrix} -\varepsilon^2, & A \\ A, & \varepsilon^3 \end{Bmatrix}$	*ade*
35	$\varepsilon^3 N_3,\ \varepsilon\ \begin{Bmatrix} -\varepsilon^3, & A \\ A, & \varepsilon^2 \end{Bmatrix}$	*bde*
36	$\varepsilon\ N_4,\ \varepsilon\ \begin{Bmatrix} -\varepsilon^4, & A \\ A, & \varepsilon \end{Bmatrix}$	*ace*
37	$K_1,\ \begin{Bmatrix} \varepsilon, & 0 \\ 0, & \varepsilon^4 \end{Bmatrix}$	*aedbc*
38	$K_2,\ \begin{Bmatrix} \varepsilon^2, & 0 \\ 0, & \varepsilon^3 \end{Bmatrix}$	*adceb*
39	$K_3,\ \begin{Bmatrix} \varepsilon^3, & 0 \\ 0, & \varepsilon^2 \end{Bmatrix}$	*abecd*
40	$K_4,\ \begin{Bmatrix} \varepsilon^4, & 0 \\ 0, & \varepsilon \end{Bmatrix}$	*acbde*
41	$\epsilon M_0,\ \epsilon \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	*adecb*
42	$\epsilon M_1,\ \epsilon \begin{Bmatrix} A, & \epsilon \\ \epsilon^4 - A & \end{Bmatrix}$	*acdeb*
43	$\epsilon M_2,\ \epsilon \begin{Bmatrix} A, & \epsilon^2 \\ \epsilon^2 - A & \end{Bmatrix}$	*acebd*
44	$\epsilon M_3,\ \epsilon \begin{Bmatrix} A, & \epsilon^3 \\ \epsilon^3 - A & \end{Bmatrix}$	*abdce*
45	$\epsilon M_4,\ \epsilon \begin{Bmatrix} A, & \epsilon^4 \\ \epsilon - A & \end{Bmatrix}$	*adcbe*
46	$\epsilon^4 M_0,\ \epsilon^4 \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	*aebcd*
47	$\epsilon^4 M_1,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon \\ \epsilon^4 - A & \end{Bmatrix}$	*abced*
48	$\epsilon^4 M_2,\ \epsilon^4 \begin{Bmatrix} a, & \epsilon^2 \\ \epsilon^3 - A & \end{Bmatrix}$	*abedc*
49	$\epsilon^4 M_3,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon^3 \\ \epsilon^2 - A & \end{Bmatrix}$	*adbec*
50	$\epsilon^4 M_4,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon^4 \\ \epsilon - A & \end{Bmatrix}$	*aecdb*
51	$\epsilon^2 N_0,\ \epsilon^2 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	*acbed*
52	$N_1,\ \begin{Bmatrix} -\epsilon, & A \\ A, & \epsilon^4 \end{Bmatrix}$	*abcde*
53	$\epsilon^3 N_2,\ \epsilon^3 \begin{Bmatrix} -\epsilon^2, & A \\ A, & \epsilon^3 \end{Bmatrix}$	*aecbd*
54	$\epsilon\ N_3,\ \epsilon\ \begin{Bmatrix} -\epsilon^3, & A \\ A, & \epsilon^2 \end{Bmatrix}$	*acdbe*
55	$\epsilon^4 N_4,\ \epsilon^4 \begin{Bmatrix} -\epsilon^4, & A \\ A, & \epsilon \end{Bmatrix}$	*abdec*
56	$\epsilon^2 N_0,\ \epsilon^2 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	*adbce*
57	$\epsilon\ N_1,\ \epsilon\ \begin{Bmatrix} -\epsilon, & A \\ A, & \epsilon^4 \end{Bmatrix}$	*aebdc*
58	$\epsilon^4 N_2,\ \epsilon^4 \begin{Bmatrix} -\epsilon^2, & A \\ A, & \epsilon^3 \end{Bmatrix}$	*acedb*
59	$\epsilon^2 N_3,\ \epsilon^2 \begin{Bmatrix} -\epsilon^3, & A \\ A, & \epsilon^2 \end{Bmatrix}$	*adebc*
60	$N_4,\ \begin{Bmatrix} -\epsilon^4, & A \\ A, & \epsilon \end{Bmatrix}$	*aedcb*

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As an example of the composition of the forms, take 56, 57, we have

$$56 = adbce.abdec,$$

$$57 = aebdc.edacb,$$

$$abcde,$$

whence $56.57 = cde, = 24,$

and we have

$$56.57 = \begin{matrix} -\varepsilon^3, & A\varepsilon^3 \\ A, & 1 \end{matrix} \left| \begin{matrix} (-\varepsilon^3, A), & (\varepsilon A, \varepsilon^4) \\ & \end{matrix} \right| = \left| \begin{matrix} 1 + A^2\varepsilon^3, & -A\varepsilon^4 + A\varepsilon^2 \\ -A\epsilon^3 + A, & A^2\epsilon + \epsilon^4 \end{matrix} \right|;$$

$$24 = \begin{matrix} A\varepsilon^3, & 1, \\ \varepsilon^3, & -A, \end{matrix}$$

viz., taking out the factor $A(\varepsilon^2 - \varepsilon^4)$, we ought to have

$$\left| \begin{matrix} 1 + A^2\varepsilon^3, & A(\epsilon^3 - \varepsilon^4) \\ A(1 - \epsilon^3), & A(\varepsilon^3 - \epsilon^4) \end{matrix} \right| = \left| \begin{matrix} A\varepsilon^3, & 1 \\ \varepsilon^3, & -A \end{matrix} \right| \left| \begin{matrix} A(\varepsilon^3 - \varepsilon^4) \\ A(\varepsilon^3 - \epsilon) \end{matrix} \right|^4 :$$

a relation which will be identically true if only

$$1 + A^2\varepsilon^3 = A^2(1 - \varepsilon^2),$$

$$\epsilon^4 + A^2\epsilon = A^2 \quad (-\epsilon^2 + \varepsilon^4),$$

these are

$$1 = A^2(\quad 1 - \epsilon^3 - \epsilon^2),$$

$$\epsilon^4 = A^2(-\varepsilon - \epsilon^2 + \varepsilon^4),$$

the second of which is obviously equivalent to the first: and this first equation is

$$1 = A^2(A + 2),$$

that is

$$A^3 + 2A^2 - 1 = 0, \text{ or } (A + 1)(A^2 + A - 1) = 0,$$

which is true in virtue of $A^2 + A - 1 = 0$; we have thus the required equation

$$56.57 = 24,$$

or say

$$\left\{ \begin{matrix} -\varepsilon^3, & A\varepsilon^3 \\ A, & 1 \end{matrix} \right\} \left\{ \begin{matrix} -\epsilon^3, & A\epsilon \\ A, & \epsilon^4 \end{matrix} \right\} = \left\{ \begin{matrix} A, & \varepsilon^2 \\ \epsilon^3, & -A \end{matrix} \right\},$$

and in like manner it may be shown that the product of any two of the sixty matrices is a matrix of the group.

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There is an interesting case of linear transformation in Gordan's paper "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Ann.*, t. 13 (1878), pp. 375–404, see p. 379. Writing as before $A = \epsilon + \epsilon^4$, $B = \epsilon^2 + \epsilon^3$, $A' = \epsilon - \epsilon^4$, $B' = \epsilon^2 - \epsilon^3$, then for y_1, y_2 we substitute $Ay_1 + y_2$, $y_1 - Ay_2$ each divided by B'; and for x_1, x_2 we make the like substitution, ϵ^2 instead of ϵ, viz., this is for x_1, x_2 $Bx_1 + x_2$, $x_1 - Bx_2$ each divided by $-A'$.

This being so, the bipartite function

$$f = (y_1^3, y_1^2y_2, y_1y_2^2, y_2^3)(x_1^3x_2, x_2^3, x_1^3, -x_1x_2^3),$$

or, what is the same thing,

$$= \begin{array}{c|cccc} & x_1^3, & x_1^2x_2, & x_1x_2^3, & x_2^3 \\ \hline y_1^3 & & 1 & & \\ y_1^2y_2 & & & & 1 \\ y_1y_2^2 & 1 & & & \\ y_2^3 & & & -1 & \end{array}$$

remains unaltered by the substitutions. Observe that we have $AB = -1$, so that omitting common factors the substitutions are

$$y_1, y_2 \text{ into } Ay_1 + y_2,\ y_1 - Ay_2,$$
$$x_1, x_2 \text{ into } -(x_1 - Ax_2),\ Ax_1 + x_2,$$

where it is to be noticed that (x_1, x_2) are neither cogredient nor contragredient with (y_1, y_2); but they are what may be termed socio-gredient, viz., the substitution for (y_1, y_2) determines uniquely that for (x_1, x_2).

The verification of the invariance of f might be effected rather more simply by means of the last mentioned forms, but it is interesting to use the original forms

$$y_1, y_2 \text{ into } Ay_1 + y_2,\ y_1 - Ay_2,$$
$$x_1, x_2 \text{ into } Bx_1 + x_2,\ x_1 - Bx_2.$$

Making the substitution, the whole coefficient of y_1^3 is found to be

$$\begin{aligned} &(A^3B^3 + A^3 + AB^3 - B)x_1^3 \\ &+ (2A^3B - A^3B^3 - 3A^2B + 3AB^2 + 2B^3 - 1)x_1^2x_2 \\ &+ (A^3 - 2A^3B^2 + 3A^2B^2 + 3AB - B^3 + 2B)x_1x_2^2 \\ &+ (-A^3B - A^2B^3 + A - B^3)x_2^3, \end{aligned}$$

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where, reducing by $AB = -1$, the several coefficients are

$$A + A^2 - B^2 - B, \quad = 0,$$
$$-2A^2 + 3A + 2B^2 - 3B, = 5(A - B),$$
$$A^2 - 2A - B^2 + 2B, \quad = 0,$$
$$A^2 - B + A - B^2, \quad = 0,$$

in virtue of the relations $A^2 + A = 1$, $B^2 + B = 1$. There are similar reductions for the coefficients of $y_1^2y_2$, $y_1y_2^2$ and y_2^3; and the whole thus becomes

$= 5(A - B)$.

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
y_1^3		1		
$y_1^2y_2$				1
$y_1y_2^2$	1			
y_2^3			-1	

but there is a denominator $-A'^3B'^3$, and we have $A'B' = B - A$, or this denominator is $(A - B)^3$, $= (A - B)(A - B)^2$, where

$$(A - B)^2 = A^2 - 2AB + B^2, = 1 - A + 2 + 1 - B, = 4 - A - B, = 5,$$

viz., the denominator is $= 5(A - B)$. Thus the new value of f is equal to its original value, which is the theorem in question.

ON SYLVESTER'S TABLES OF HAMILTONIAN DIFFERENCES AND THEIR ASSOCIATE NUMBERS.

By J. C. GLASHAN, Ottawa, Canada.

IN his "Note on Hamilton's Numbers," in Vol. XXVII. of this *Journal*, pp. 184–188, Professor Mathews attributed to the late M. Edward Lucas the table of numbers which he gives on page 185 of his Note, but this would appear to be a mistake. The table—and, in point of fact, the whole investigation—contained in Note 4 of Lucas' *Théorie des Nombres*, pp. 495–498, are due to Professor Sylvester. (See *Sur les Nombres dits de Hamilton, par J. J. Sylvester, Association française pour l'avancement des Sciences, compte rendu de la 16me session, Toulouse*, 1887, 2e *partie*, pp. 164–167. See also *Note on Hamilton's Numbers, by J. J. Sylvester, Nature*, Vol. XXXVI., p. 557, 558). While restoring both table and investigation to Professor Sylvester, it is but just to point out that

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M. Lucas lays no claim to them himself, but gives a reference to Sylvester's paper in the Reports of the Association française, and to Sylvester and Hammond's memoirs in the *Philosophical Transactions*, Vol. CLXXVIII., pp. 285–312, and Vol. CLXXIX., pp 65–71.

The change in the terms in the hypothenuse of Sylvester's second or longer table which Professor Mathews makes on p. 186 of his Note, although slight, is important. By it all the rows, and consequently all the columns, of the table become pure series of differences, and by extending the line series to the left as in the following table, the complete law of relation of the terms in all these series becomes apparent, and the general term may be written down at once. The following is a portion of the table thus extended:

	$(n,-1)$	$(n,0)$	$(n,1)$	$(n,2)$	$(n,3)$	$(n,4)$	$(n,5)$	$(n,6)$	...
$(0,n)$		1	[1]	1	1	1	1	1	...
$(1,n)$		0	1	[2]	3	4	5	6	...
$(2,n)$	−1	−1	0	2	5	9	14	20	...
$(3,n)$	0	−1	−1	1	[6]	15	29	49	...
$(4,n)$	1	0	−1	0	6	21	50	99	...
$(5,n)$		0	−1	−1	5	26	76	175	...
$(6,n)$		1	0	−1	4	30	106	281	...
$(7,n)$			0	−1	3	33	139	420	...
$(8,n)$			0	−1	2	35	174	594	...
$(9,n)$			0	−1	1	[36]	210	804	...
$(10,n)$			0	−1	0	36	246	1050	...
$(11,n)$			0	−1	−1	35	281	1331	...
$(12,n)$			1	0	−1	34	315	1646	...
$(13,n)$				0	−1	33	348	1994	...
				⋮		⋮	⋮	⋮	
$(44,n)$				0	−1	2	875	23694	...
$(45,n)$				0	−1	1	[876]	24570	...
$(46,n)$				0	−1	0	876	25446	...
$(47,n)$				0	−1	−1	875	26321	...
$(48,n)$				1	0	−1	874	27195	...
⋮					⋮	⋮	⋮	⋮	

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The terms enclosed in square brackets are the first differences of the successive Hamiltonian numbers, and Sylvester has for that reason named them Hamiltonian Differences and also Hypothenusal Numbers, they being the numbers in the hypothenuse of his first or smaller table of Hamiltonian Differences and their associate numbers.

If $h_r \equiv$ the r^{th} Hamiltonian difference,

$$S_r \equiv h_1 + h_2 + h_3 + \ldots + h_r,\ S_0 = 0,$$

$$\binom{a}{c} \equiv \frac{a!}{c!(a-c)!},$$

and if (m, n) denote the term which is in the $(1+m)^{th}$ line and the $(2+n)^{th}$ column of the table, the general expression for any term will be

$$(m, n) = \binom{m+n-1}{m} - \binom{m+n-2}{m-2} + \binom{m+n-S_1-3}{m-S_1-3} + \binom{m+n-S_2-4}{m-S_2-3} + \binom{m+n-S_3-5}{m-S_3-3} + \binom{m+n-S_4-6}{m-S_4-3} + \ldots (A).$$

Also $(S_r, r+1) \equiv h_{r+1}$,

therefore

$$h_{r+1} = \binom{r+S_r}{S_r} - \binom{r+S_r-1}{S_r-2} + \binom{r+S_r-S_1-2}{S_r-S_1-3} + \binom{r+S_r-S_2-3}{S_r-S_2-3} + \binom{r+S_r-S_3-4}{S_r-S_3-3} + \ldots + \binom{h_r}{h_r-3} \ldots (B).$$

By formula (B) the Hamiltonian differences may be computed each in succession from those that go before, and $S_1, S_2, S_3, \ldots$ may therefore be considered in (A) as given numbers.

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To adapt (*A*) to Professor Mathews' table

1	1	1	1	1	1	...
1	2	3	4	5	6	...
	2	5	9	14	20	...
	1	6	15	29	49	...
		6	21	50	99	...
		5	26	76	175	...
		4	30	106	281	...
		3	33	139	420	...
		2	35	174	594	...
		1	36	210	804	...
			36	246	1050	...
			35	281	1331	...
			⋮	⋮	⋮	

let $(m, t)_r$ denote the t^{th} term in the $(1+m)^{\text{th}}$ line, this line commencing at the r^{th} column of the table, and therefore $S_{r-2} < m < S_{r-1} + 1$, then will

$$(m, t)_r = \binom{m+r+t-3}{m} - \binom{m+r+t-4}{m-2}$$
$$+ \binom{m+r+t-S_1-5}{m-S_1-3} + \binom{m+r+t-S_2-6}{m-S_2-3} + \ldots (C).$$

Formulæ for the term of the first seven lines of his table have been given by Professor Mathews. Following the above general formula these may be written

$$(0, t)_1 = 1,$$

$$(1, t)_2 = \binom{t}{1} = t,$$

$$(2, t)_3 = \binom{t+2}{2} - \binom{t+1}{1},$$

$$(3, t)_3 = \binom{t+3}{3} - \binom{t+2}{1},$$

$$(4, t)_4 = \binom{t+5}{4} - \binom{t+4}{2} + \binom{t+2}{0},$$

$$(5, t)_4 = \binom{t+6}{5} - \binom{t+5}{3} + \binom{t+3}{1},$$

$$(6, t)_4 = \binom{t+7}{6} - \binom{t+6}{4} + \binom{t+4}{2} + \binom{t+1}{0}.$$

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The series (S_0, n), (S_1, n), (S_2, n), (S_3, n), ... constitute Sylvester's first or shorter table, which might well be called Sylvester's triangle (of Hamiltonian Differences and their Associate Numbers), and of which the following is the beginning:

1	1	1	1	1	1	...
	2	3	4	5	6	...
		6	15	29	49	...
			36	210	804	...
				876	24570	...
					408696	...
					⋮	

If $[r, t]$ denote the t^{th} term of the r^{th} line of this table, it follows from the general formula for the terms of the larger table that

$$[r+1, t] = \binom{S_r + r + t - 1}{S_r} - \binom{S_r + r + t - 2}{S_r - 2} + \binom{S_r + r + t - S_1 - 3}{S_r - S_1 - 3} + \binom{S_r + r + t - S_2 - 4}{S_r - S_2 - 3} + \ldots + \binom{S_r + t - S_{r-1} - 1}{S_r - S_{r-1} - 3} \ldots (D).$$

The terms of the first four lines of this shorter table are therefore

$$[1, t] = \binom{t-1}{0} = 1,$$

$$[2, t] = \binom{t+1}{1} = t + 1,$$

$$[3, t] = \binom{t+4}{3} - \binom{t+3}{1},$$

$$[4, t] = \binom{t+11}{9} - \binom{t+10}{7} + \binom{t+8}{5} + \binom{t+5}{3}.$$

These formulæ may be compared with those given by Professor Mathews on p. 188.

"The question arises as to whether it is possible to deduce the Hamiltonian Differences, or to reduce the Hamiltonian Numbers, directly in a continued chain from one another

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without the use of any intermediate numbers. Mr. James Hammond has shown that it is possible, and has made the remarkable discovery that it is the Numbers of Hamilton, and not the Hypothenusal Numbers, which are subject to a very simple scale of relation. This being found, of course the differences become known. This is contrary to what one would have expected. *A priori*, one would have anticipated that the determination of the Hypothenusal Numbers would have preceded that of their sums." (*J. J. Sylvester*, in *Philosophical Transactions*, Vol. 178, p. 289).

Professor Sylvester felt that there must be a simply demonstrated, a natural, method of computing the Hamiltonian differences 'in a continued chain from one another,' and Professor Mathews, by the change which he has introduced into Sylvester's longer table, has justified that feeling. That change not only made actually self-evident the simple scale of relation among the Hamiltonian differences which has been given in (*B*), but it made apparent quite as simple a law of relation to these differences of all the numbers in the table, viz. the relation expressed by (*A*).

ON TWISTED QUARTICS OF THE SECOND SPECIES.

By A. R. Forsyth.

1. MOST of the properties of a twisted quartic are usually deduced from the consideration that it is the intersection of a quadric either with another quadric or with a cubic surface. When it is the intersection of two quadrics, it is said to be of the first species and it constitutes the whole intersection of the surfaces; when it is the intersection of a quadric and a cubic, it is said to be of the second species and it forms only part of the intersection of the surfaces, the remainder consisting of two right lines that do not meet. The processes adopted to derive the properties are often of an entirely synthetic character.*

* For twisted quartics of the first species, see Schroeter, *Grundzüge einer rein-geometrischen Theorie der Raumkurve vierter Ordnung erster Species* (Leipzig, 1890), where many references are given; for those of the second species, see Cremona, *Intorno alla curva gobba del quart' ordine per la quale passa una sola superficie di secondo grado*, Ann. di Mat., t. 4 (1861), pp. 71—101, where references to Plücker, Cayley, Salmon and Steiner are given.

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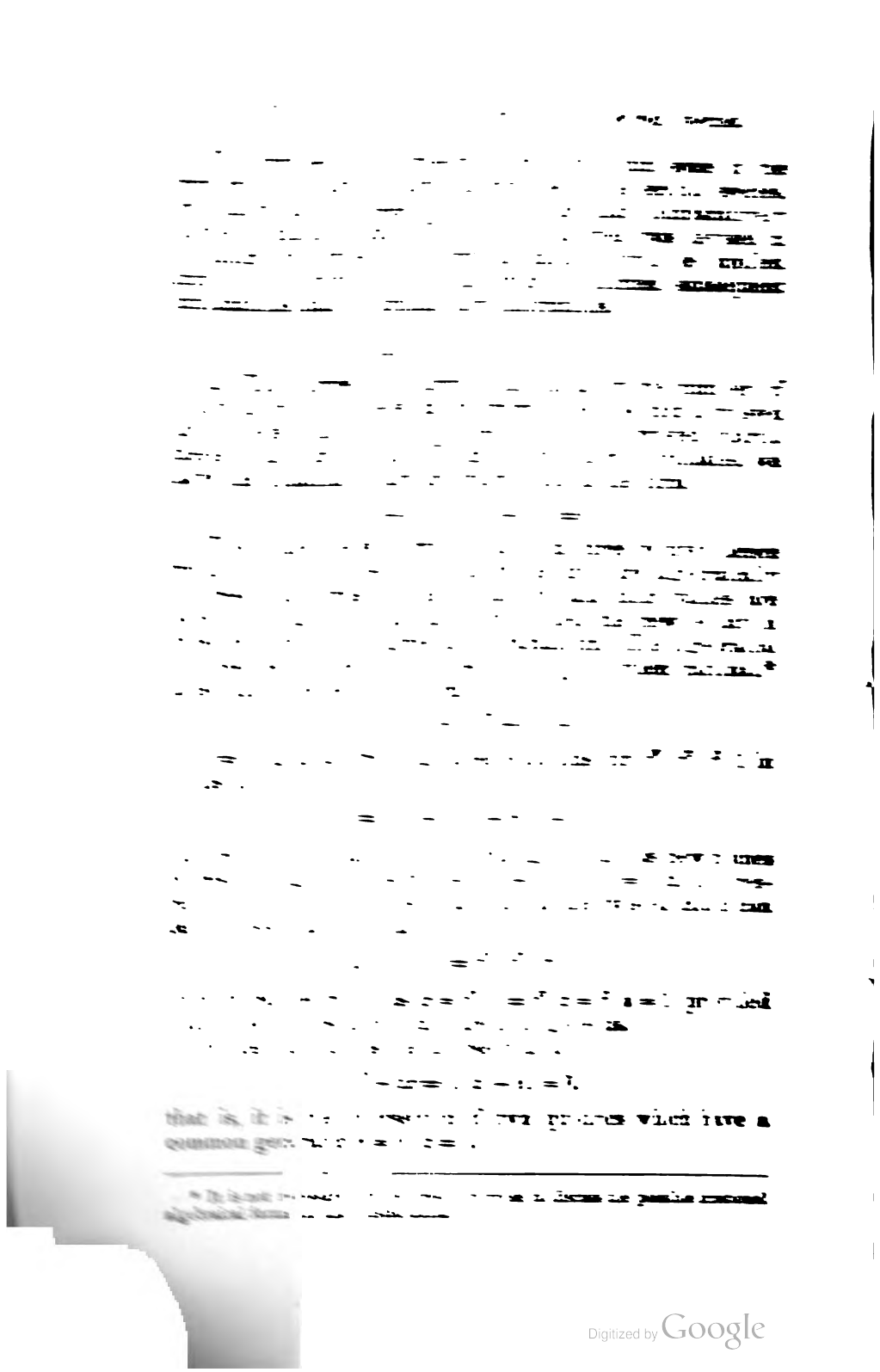

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3. The cubic must meet any plane

$$px + qy + rz + su = 0$$

in three points; if a, b, c be the parameters of these points, they are the roots of

$$p\theta^3 + q\theta^2 + r\theta + s = 0.$$

Consequently, the equation of the plane through three points a, b, c on the cubic is

$$x - (a + b + c)y + (ab + bc + ca)z - abcu = 0.$$

The equation of the osculating plane at a is

$$x - 3ay + 3a^2z - a^3u = 0.$$

The equations of the tangent line at a are those of the line common to the osculating planes at a and at $a + \delta a$ respectively; or they are those of the line common to the planes through a, a, c and a, a, d. These equations are

$$x - 2ay + a^2z = 0, \quad y - 2az + a^2u = 0.$$

Any plane through the tangent at a is

$$x - 2ay + a^2z - \lambda(y - 2az + a^2u) = 0;$$

it meets the curve again in the point that has λ for its parameter.

If the osculating plane at θ pass through the point X, Y, Z, U in space, then

$$X - 3\theta Y + 3\theta^2 Z - \theta^3 U = 0.$$

Evidently three osculating planes will in general pass through such a point; let them osculate at a, b, c respectively, so that a, b, c are the roots of the preceding equation. Hence the coordinates of the point of intersection of the three osculating planes at a, b, c are determined by the equations

$$\frac{X}{abc} = \frac{3Y}{ab + bc + ca} = \frac{3Z}{a + b + c} = U;$$

and the point X, Y, Z, U therefore lies in the plane

$$x - (a + b + c)y + (ab + bc + ca)z - abcu = 0,$$

which is the plane through the three points. Hence it follows that, through any point in space, three osculating planes can

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be drawn to a cubic: and the plane through the three osculations contains the point. The point is called the *focus* of the plane; hence the plane which has X, Y, Z, U for focus is

$$xU - 3yZ + 3zY - uX = 0.$$

The condition that a point X', Y', Z', U' should lie in this plane is

$$X'U - 3Y'Z + 3Z'Y - U'X = 0,$$

which is the condition that X, Y, Z, U should lie in the plane

$$xU' - 3yZ' + 3zY' - uX' = 0:$$

and this plane has X', Y', Z', U' for its focus. Hence if a point A lie in a plane whose focus is B, the point B lies in the plane whose focus is A.

Any point on the line AB is $\lambda X + \mu X'$, $\lambda Y + \mu Y'$, $\lambda Z + \mu Z'$, $\lambda U + \mu U'$; the plane, of which this is the focus, is

$$x(\lambda U + \mu U') - 3y(\lambda Z + \mu Z') + 3z(\lambda Y + \mu Y') - u(\lambda X + \mu X') = 0$$

which, for all values of $\lambda : \mu$, passes through the line which is the intersection of the planes having A and B for focus respectively. There is thus a reciprocal relation between a point and a plane, and between a line and a line.*

4. Because the plane through the three points a, b, c is

$$x - (a + b + c)y + (ab + bc + ca)z - abcu = 0,$$

it follows that the line

$$x - (a + b)y + abz = 0, \quad y - (a + b)z + abu = 0$$

meets the cubic in a and b. Hence through any point X, Y, Z, U in space a single line can be drawn to meet the cubic in two points; and the two points are determined by the equations

$$\frac{ab}{Y^2 - XZ} = \frac{a + b}{YZ - XU} = \frac{1}{Z^2 - YU}.$$

The cone having its vertex at X, Y, Z, U (supposed not to lie on the curve), and passing through the curve, is a cubic

* This proof is practically the same as that given in Salmon's *Solid Geometry*, 3rd edition, p. 307.

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cone; and the line through $XYZU$ meeting the curve in a and b is a double generator of the cone. When a plane section is taken of this cone, we have a cubic curve; and the intersection of the plane and the double generator is a double point of the curve. Hence a twisted cubic projects conically on any plane into a cubic with a double point.

All these propositions are known*; as already remarked, they are proved again merely to illustrate the method of proceeding and to anticipate explanations in the case of twisted quartics to which we now proceed.

Application to Quartics.

5. Any arbitrary plane

$$l_1y_1 + l_2y_2 + l_3y_3 + l_4y_4 = 0$$

meets a twisted quartic in four points; hence when the coordinates of any point on the quartic are expressed algebraically in terms of one parameter, and their values are substituted in the equation of the plane, the new equation must lead to four values of the parameter.

If all the coordinates y_1, y_2, y_3, y_4 be irrational in the parameter, then in order that the equation of every plane may lead to four values only, we must have $y_1^2 = a'\theta + a$, $y_2^2 = b'\theta + b$, $y_3^2 = c'\theta + c$, $y_4^2 = d'\theta + d$; the curve is then the common intersection of the cluster of quadrics

$$\begin{Vmatrix} y_1^2, & y_2^2, & y_3^2, & y_4^2 \\ a', & b', & c', & d' \\ a, & b, & c, & d \end{Vmatrix} = 0:$$

it is therefore a quartic of the first species.

If three of the coordinates be irrational in the parameter, and the other coordinate rational, we must have expressions of the form

$$y_1^2 = a'\theta + a,\ y_2^2 = b'\theta + b,\ y_3^2 = c'\theta + c,\ y_4 = d;$$

this is a special case of the former, obtained by making d' zero.

If two of the coordinates be irrational and the other two

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* Most of them occur in Cremona's memoir, *Crelle*, Vol. LVIII., (1861), pp. 138—151; others there given can be deduced in a similar manner. See also Salmon's *Solid Geometry* (3rd edition), pp. 305—308.

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rational in the parameter, we must have expressions of the form

$$y_1^2 = a\theta^2 + 2a\theta + b,\ y_2^2 = \beta\theta^2 + 2a'\theta + b',$$

$$y_3 = \gamma\theta + \delta,\ \ y_4 = \gamma'\theta + \delta' :$$

so that taking

$$\delta' y_3 - \delta y_4 = (\gamma\delta' - \gamma'\delta)\, x,$$

$$-\gamma' y_3 + \gamma y_4 = (\gamma\delta' - \gamma'\delta)\, u,$$

we have

$$x = \theta,\ y = u.$$

The curve is then the intersection of the quadrics

$$\left.\begin{aligned} y_1^2 &= ax^2 + 2axu + bu^2 \\ y_2^2 &= \beta x^2 + 2a'xu + b'u^2 \end{aligned}\right\} :$$

it is therefore a quartic of the first species. A very particular case arises when

$$y_1 = \operatorname{cn}\phi,\ y_2 = \operatorname{dn}\phi,\ y_3 = \operatorname{sn}\phi,\ y_4 = 1 :$$

but in this rational form, the coordinates are not expressed algebraically in terms of the parameter ϕ.

If one of the coordinates be irrational and the remaining three be rational in the parameter, we must have

$$y_r = a_r\theta^2 + b_r\theta + c_r,\ \ (r = 1, 2, 3)$$

$$y_4^2 = a\theta^4 + \beta\theta^3 + \gamma\theta^2 + \delta\theta + \varepsilon.$$

From the first three we have

$$\frac{\theta^2}{a_1 y_1 + a_2 y_2 + a_3 y_3} = \frac{\theta}{\beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3} = \frac{1}{\gamma_1 y + \gamma_2 y_2 + \gamma_3 y_3},$$

say

$$\frac{\theta^2}{x} = \frac{\theta}{z} = \frac{1}{u}.$$

The curve is then the intersection of

$$z^2 = xu,$$

$$y_4^2 = ax^2 + \beta xz + \gamma z^2 + \delta zu + \varepsilon u^2;$$

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it is therefore a quartic of the first species. A very particular case arises when

$$y_1 = P^2(\mu),\ y_2 = P(\mu),\ y_3 = 1,\ y_4 = P'(\mu);$$

but again the coordinates are not expressed rationally in terms of μ.

In every case therefore in which one or more of the coordinates of a point on a twisted quartic, supposed expressed algebraically in terms of one parameter, are irrational in their expressions, the quartic is of the first species. References to such quartics, and in particular to the association of doubly periodic functions with them as effected by Clebsch, Harnack and others, are given in Schroeter's monograph.*

The remaining case, being the only one in which twisted quartics of the second species can occur, arises when all the coordinates are rational in the parameter. As the equation is to be of the fourth degree when they are substituted in

$$l_1 y_1 + l_2 y_2 + l_3 y_3 + l_4 y_4 = 0,$$

we must in the most general case have expressions

$$y_r = a_r \theta^4 + b_r \theta^3 + c_r \theta^2 + d_r \theta + e_r,$$

for $r = 1, 2, 3, 4$. Solving these equations for θ^4, θ^3, θ^2 and θ in terms of y_1, y_2, y_3, y_4 and the constants, we have

$$\theta^s = \alpha_s y_1 + \beta_s y_2 + \gamma_s y_3 + \delta_s y_4 + \varepsilon_s$$

for $s = 1, 2, 3, 4$. Let x, y, z, u be new planes corresponding to $\alpha_s y_1 + \beta_s y_2 + \gamma_s y_3 + \delta_s y_4$ for $s = 4, 3, 2, 1$ respectively; the coordinates of a point on the twisted quartic can be expressed in the form

$$x : y : z : u = \theta^4 + \alpha : \theta^3 + \beta : \theta^2 + \gamma : \theta + \delta,$$

or, as all the equations in x, y, z, u are homogeneous, *the coordinates of a point on the twisted quartic can be taken in the form*

$$x = \theta^4 + \alpha,\ y = \theta^3 + \beta,\ z = \theta^2 + \gamma,\ u = \theta + \delta.$$

Manifestly this is not the only method in which the equations expressing y_1, y_2, y_3, y_4 in terms of θ can be used.

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* See note, p. 247.

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Thus it is easy to see that the coordinates of a point on the twisted quartic can be taken in the form

$$x' = \theta^4 + \alpha'\theta^3,\ y' = \theta^3 + \beta'\theta^2,\ z' = \theta^2 + \gamma'\theta,\ u = \theta + \delta,$$

and then we have

$$x' = x + \alpha' y,\ y' = y + \beta' z,\ z' = z + \gamma' u,$$
$$\alpha = -\alpha'\beta'\gamma'\delta,\ \beta = \beta'\gamma'\delta,\ \gamma = -\gamma'\delta.$$

The former expressions will be used.

It cannot, of course, be inferred that a quartic determined by

$$x : y : z : u = \theta^4 + \alpha : \theta^3 + \beta : \theta^2 + \gamma : \theta + \delta$$

is necessarily of the second species; the preceding investigations shew all quartics of the second species can be represented in this form. In what follows, the discussion will be limited to quartics which can thus be represented.

6. It is known that one quadric can be drawn through a twisted quartic; and that the quartic is of the second species when only a single quadric can be drawn through it. The equation of this quadric must be homogeneous of the second order; and in the case of each species of quartic the equation must be satisfied identically when

$$x = \theta^4 + \alpha,\ y = \theta^3 + \beta,\ z = \theta^2 + \gamma,\ u = \theta + \delta$$

are substituted. This condition supplies nine equations which potentially are sufficient to obtain the nine independent coefficients for the quadric.

The slightest consideration shews that the terms in x^2 and xy must be absent; that xz and y^2 occur in the combination $xz - y^2$, and that xu and yz occur in the combination $xu - yz$. Taking the equation of the quadric in the form

$$v(xz - y^2) + t(xu - yz) + sz^2 + ryu + qzu + pu^2 = 0,$$

the ratios of the constants v, t, s, r, q, p are given by the equations

$$\left.\begin{array}{lllllll}
v\gamma & +t\delta & +s & +r & & & =0 \\
-v2\beta & -t\gamma & & +r\delta & +q & & =0 \\
v\alpha & -t\beta & +s2\gamma & & +q\delta & +p & =0 \\
 & t\alpha & & +r\beta & +q\gamma & +p2\delta & =0 \\
v(\alpha\gamma - \beta^2) & +t(\alpha\delta - \beta\gamma) & +s\gamma^2 & +r\beta\delta & +q\gamma\delta & +p\delta^2 & =0
\end{array}\right\}.$$

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When these equations are solved to determine the five ratios, the result is

$$0 = \quad v(\beta + \gamma\delta) J + t(\gamma + \delta^2) J,$$

$$0 = \quad v(\alpha + \beta\delta) J + s(\gamma + \delta^2) J,$$

$$0 = -v(\alpha - \gamma^2 + 2\beta\delta) J + r(\gamma + \delta^2) J,$$

$$0 = \quad v(\alpha\delta - \beta\gamma) J + q(\gamma + \delta^2) J,$$

$$0 = -v(\alpha\gamma - \beta^2) J + p(\gamma + \delta^2) J,$$

where J denotes

$$\alpha(\delta^2 + \gamma) - \beta^2 - 2\beta\gamma\delta + \gamma^2.$$

If J does not vanish the coefficients are uniquely determined; the equation of the quadric takes the form

$$(\gamma + \delta^2)(xz - y^2) + (\beta + \gamma\delta)(yz - xu) - (\alpha + \beta\delta) z^2$$
$$+ (\alpha - \gamma^2 + 2\beta\delta) yu + (\beta\gamma - \alpha\delta) zu + (\alpha\gamma - \beta^2) u^2 = 0,$$

or, what is the same thing,

$$V = \begin{vmatrix} y, & x + \beta u, & \alpha + \beta\delta \\ z, & y + \gamma u, & \beta + \gamma\delta \\ u, & z + \delta u, & \gamma + \delta^2 \end{vmatrix} = 0.$$

Only a single quadric can be drawn through the curve in this case; the quartic is of the second species.

But, if J does vanish the five equations are not independent of one another; one of them can be deduced from the other four. In this case the four equations determine p, q, r, s homogeneously in terms of t and v, but there is no relation to determine $t : v$. The quadric is not determinate; there is a cluster of quadrics passing through the curve which is the complete intersection of two quadrics of the form

$$U_1 = xz - y^2 + s_1 z^2 + r_1 yu + q_1 zu + p_1 u^2,$$

$$U_2 = xu - yz + s_2 z^2 + r_2 yu + q^2 zu + p_2 u^2,$$

the cluster being

$$vU_1 + tU_2 = 0.$$

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[redacted]

[redacted]

[redacted]

in four points; if a, b, c, d be the parameters of these points, they are the roots of

$$l\theta^4 + m\theta^3 + n\theta^2 + p\theta + l\alpha + m\beta + n\gamma + p\delta = 0,$$

so that

$$-m = l(a+b+c+d),$$

$$n = l\{ab+bc+ca+d(a+b+c)\},$$

$$-p = l\{abc+d(ab+bc+ca)\}.$$

and $$labcd = l\alpha + m\beta + n\gamma + p\delta.$$

Now any plane is determined by three points in general, if in the present case these be the points a, b, c, the remaining point d in which the plane cuts the curve is given by

$$abcd = \alpha - \beta(a+b+c+d) + \gamma\{ab+bc+ca+d(a+b+c)\} - \delta\{abc+d(ab+bc+ca)\},$$

so that it is

$$d = \frac{\alpha - \beta(a+b+c) + \gamma(ab+bc+ca) - \delta abc}{\beta - \gamma(a+b+c) + \delta(ab+bc+ca) + abc},$$

and this is the condition that *the four points* a, b, c, d *lie in one plane.*

Hence the equation of the plane through the three points a, b, c, is

$$x - y(a+b+c+d) + z\{ab+bc+ca+d(a+b+c)\} - u\{abc+d(ab+bc+ca)\} = 0,$$

where d is the parameter of the fourth point in which the plane cuts the curve, its value in terms of a, b, c being given by the preceding expression.

8. Let the osculating plane at a cut the curve again in a', so that a' can be regarded as a residual of a: we have

$$a' = \frac{\alpha - 3\beta a + 3\gamma a^2 - \delta a^3}{\beta - 3\gamma a + 3\delta a^2 + a^3}.$$

The equation of the osculating plane at a is

$$x - (3a + a')y + (3a^2 + 3aa')z - (a^3 + 3a^2a')u = 0,$$

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which on substitution for a' becomes

$$(x - 3ay + 3a^2z - a^3u)(\beta - 3a\gamma + 3a^2\delta + a^3)$$
$$= (y - 3az + 3a^2u)(\alpha - 3a\beta + 3a^2\gamma - a^3\delta).$$

Hence,* *through any point in space six osculating planes can be drawn to a twisted quartic of the second species.*

If $a' = a$, the plane contains four consecutive points of the curve; at such a point the sphere of curvature becomes a plane. The parameter a is determined by the equation

$$a^4 + 4a^3\delta - 6a^2\gamma + 4a\beta - \alpha = 0,$$

and the plane at the point is

$$x - 4ay + 6a^2z - 4a^3u = 0.$$

There are evidently *four points of triple plane contact*: at each of them the radius of spherical curvature is infinite.

Let I and J denote the quadrinvariant and the cubinvariant of

$$a^4 + 4a^3\delta - 6a^2\gamma + 4a\beta - \alpha = 0,$$

so that

$$I = -\alpha - 4\beta\delta + 3\gamma^2,$$

$$J = \alpha(\gamma + \delta^2) - \beta^2 - 2\beta\gamma\delta + \gamma^3.$$

Then $J = 0$ is the condition that the quartic should be of the first species. Also

$$I^3 - 27J^2 = 0$$

is the condition that two of the roots a should coincide, that is, that there should be only three points of triple plane contact; and $I = 0$, $J = 0$ is the condition that there should be only two, this being secured by the coincidence of three of the points.

Now the condition that four points taken anywhere on the curve should lie in a plane is

$$abcd = \alpha - \beta\Sigma a + \gamma\Sigma ab - \delta\Sigma abc;$$

and therefore the four points of triple plane contact lie in a plane if

$$-\alpha = \alpha + 4\beta\delta - 6\gamma^2 + 4\beta\delta,$$

i.e. if

$$I = 0.$$

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* Cremona, l.c., p. 87.

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Hence, *for quartics of the second species such that*

$$\alpha + 4\beta\delta - 3\gamma^2 = 0,$$

the four points of triple plane contact are coplanar: but they are not coplanar if this condition be not satisfied.

For quartics of the first species $J = 0$, so that the four roots are in harmonic ratio; if, in addition, $I = 0$ there are only two points. Hence *quartics of the first species, represented by*

$$x : y : z : u = \theta^4 + \alpha\,;\ \theta^3 + \beta\,;\ \theta^2 + \gamma : \theta + \delta,$$

for which $I = 0$, possess only two points of triple plane contact.

9. Again, the equation

$$a' = \frac{\alpha - 3\beta a + 3\gamma a^2 - \delta a^3}{\beta - 3\gamma a + 3\delta a^2 + a^3}$$

may be considered as determining a in terms of a': hence through any point of the quartic three planes can be drawn osculating the curve elsewhere. If the points of osculation have a, b, c for their parameters, a, b, c are the roots of

$$\theta^3(a' + \delta) + 3\theta^2(a'\delta - \gamma) - 3\theta(a'\gamma - \beta) + a'\beta - \alpha = 0\,;$$

so that

$$\begin{aligned}(a' + \delta)(a + b + c) &= -3a'\delta + 3\gamma,\\ (a' + \delta)(ab + bc + ca) &= -3a'\gamma + 3\beta,\\ (a' + \delta)\,abc &= -a'\beta + \alpha.\end{aligned}$$

Hence

$$\begin{aligned}\alpha - \beta(a + b + c + a') + \gamma\{ab + bc + ca + a'(a + b + c)\}\\ - \delta\{abc + a'(ab + bc + ca)\} = a'abc\,;\end{aligned}$$

and therefore (§ 7) the four points a', a, b, c lie in one plane. The equation of this plane is

$$\begin{aligned}x - y(a + b + c + a') + z\{ab + bc + ca + a'(a + b + c)\}\\ - u\{abc + a'(ab + bc + ca)\} = 0\end{aligned}$$

which becomes

$$\begin{aligned}&\delta x - 3\gamma y + 3\beta z - \alpha u\\ &+ a'(x + 2\delta y - 2\beta u)\\ &+ a'^2(-y - 3\delta z + 3\gamma u) = 0,\end{aligned}$$

on substitution for $a + b + c$, $ab + bc + ca$, abc in terms of a'.

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The envelope of this plane is

$$(x + 2\delta y - 2\beta u)^2 + 4(\delta x - 3\gamma y + 3\beta z - \alpha u)(y + 3\delta z - 3\gamma u) = 0,$$

a quadric cone having its vertex at the point

$$\left.\begin{aligned} x + 2\delta y - 2\beta u &= 0 \\ \delta x + 3\gamma y + 3\beta z - \alpha u &= 0 \\ y + 3\delta z - 3\gamma u &= 0 \end{aligned}\right\}.$$

Hence it follows* that *through any point O on a quartic three planes can be drawn osculating the curve elsewhere: the three points of osculation lie in a plane passing through O: and the envelope of this plane is a quadric cone.*

10. Let a plane, drawn through the tangent at a, meet the curve elsewhere in c and d; then since c, d, and a repeated lie in one plane

$$\alpha - 2a\beta + \gamma a^2 - (c + d)(\beta - 2a\gamma + a^2\delta) + cd(\gamma - 2a\delta - a^2) = 0.$$

The equation of the plane is

$$x - 2ay + a^2z - (c + d)(y - 2az + a^2u) + cd(z - 2au) = 0.$$

The tangent line can be regarded as the intersection of this plane and another plane passing through the tangent and meeting the curve elsewhere in c' and d'; equations similar to the above determine this plane. *The equations of the tangent line at a* are therefore

$$\frac{x - 2ay + a^2z}{\alpha - 2a\beta + a^2\gamma} = \frac{y - 2az + a^2u}{\beta - 2a\gamma + a^2\delta} = \frac{z - 2au}{\gamma - 2a\delta - a^2}.$$

Among the planes through this line, we select those which meet the curve elsewhere in two coincident points θ and which therefore contain the tangent at θ. The point θ is determined as a root of the quadratic

$$\alpha - 2\beta(a + \theta) + 4\gamma a\theta + \gamma(a^2 + \theta^2) - 2\delta a\theta(a + \theta) - a^2\theta^2 = 0,$$

and the equation of the plane is

$$x - 2y(a + \theta) + z(a^2 + 4a\theta + \theta^2) - 2u(a^2\theta + a\theta^2) = 0.$$

There are evidently two planes through a tangent at α containing a tangent at another point of the curve. Also the

* Cremona, l.c., p 98.

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quadratic is symmetric in θ and a; hence if b and c be its roots when θ is regarded as the quantity to be determined, then c and a are the roots when a is regarded as the quantity to be determined. It therefore follows* that *through the tangent at a two planes can be drawn each containing the tangent at some other point; if the two points be at b and c, the tangents at b and c lie in a plane.* Each such plane is a double tangent plane.

Next, let X, Y, Z, U be a point in space; then if a double tangent plane can be drawn through it, containing the tangents at a, θ, these parameters are determined by the equations

$$\alpha - 2\beta(a+\theta) + 4\gamma a\theta + \gamma(a^2+\theta^2) - 2\delta a\theta(a+\theta) - a^2\theta^2 = 0,$$

$$X - 2Y(a+\theta) + Z(a^2 + 4a\theta + \theta^2) - 2Ua\theta(a+\theta) = 0.$$

These equations give four systems of values; and therefore *through any point in space four double tangent planes can be drawn to a twisted quartic of the second species.*†

Let any line be given by

$$\left.\begin{aligned} lx + my + nz + pu &= 0 \\ l'x + m'y + n'z + p'u &= 0 \end{aligned}\right\}.$$

When the equations of the tangent are taken in the form

$$\left.\begin{aligned} x - 2ay + a^2z &= \theta(\alpha - 2a\beta + a^2\gamma) \\ y - 2az + a^2u &= \theta(\beta - 2a\gamma + a^2\delta) \\ z - 2au &= \theta(\gamma - 2a\delta - a^2) \end{aligned}\right\},$$

it is easy to shew that, in order that the tangent and the line may meet, the condition

$$(4a^3l + 3a^2m + 2an - p)$$

$$\{l'(\alpha - 4a^3\delta - 3a^4) + m'(\beta - 3a^2\delta - 2a^3) + n'(\gamma - 2a\delta - a^2)\}$$

$$= (4a^3l' + 3a^2m' + 2an' - p')$$

$$\{l(\alpha - 4a^3\delta - 3a^4) + m(\beta - 3a^2\delta - 2a^3) + n(\gamma - 2a\delta - a^2)\}$$

must be satisfied. This equation is of the sixth degree in a; and therefore *any straight line meets six tangents of the quartic,*

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* Cremona, l.c., p. 91. † Cremona, l.c., p. 94.

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or, what is an equivalent statement, *the osculating developable of the quartic is of the sixth order.**

11. Let a chord be drawn joining a and b: and through it draw any plane meeting the curve elsewhere say in c and d. The equation of the plane is

$$x - y\theta + z\phi - (c+d)(y - z\theta + u\phi) + cd(z - u\theta) = 0,$$

where

$$\alpha - \beta\theta + \gamma\phi - (c+d)(\beta - \gamma\theta + \delta\phi) + cd(\gamma - \delta\theta - \phi) = 0$$

and $\theta = a + b$, $\phi = ab$. Hence, as in the case of the tangent, it follows that *the equations of the chord* are

$$\frac{x - y\theta + z\phi}{\alpha - \beta\theta + \gamma\phi} = \frac{y - z\theta + u\phi}{\beta - \gamma\theta + \delta\phi} = \frac{z - u\theta}{\gamma - \delta\theta - \phi}.$$

Let X, Y, Z, U be any point in space; the lines that can be drawn through it to meet the quartic twice are the chords joining a and b, where $a + b$ and ab are determined by the equations

$$\frac{X + Y\theta + Z\phi}{\alpha - \beta\theta + \gamma\phi} = \frac{Y - Z\theta + U\phi}{\beta - \gamma\theta + \delta\phi} = \frac{Z - U\theta}{\gamma - \delta\theta - \phi}.$$

Let λ be the common value of the fractions; when λ is known, then θ and ϕ are determined uniquely, and therefore there are as many chords through X, Y, Z, U as there are values of λ. It is easy to see that λ is determined by the equation

$$\begin{vmatrix} x - \lambda\alpha, & y - \lambda\beta, & z - \lambda\gamma \\ y - \lambda\beta, & z - \lambda\gamma, & u - \lambda\delta \\ z - \lambda\gamma, & u - \lambda\delta, & \lambda \end{vmatrix} = 0,$$

which is a cubic, so that in general there are three values.

But when the quartic is of the first species the coefficient of λ^3 vanishes, so that then there are only two values.

Hence† *through any point in space three chords of a quartic of the second species can be drawn and two chords of a quartic of the first species can be drawn.*

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* Cremona, l.c., p. 86. † Cremona, l.c., p. 94; Schroeter, l.c., p. 9.

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12. We can now obtain a cubic surface through the quartic. Let a cone be drawn having any point a on the curve for vertex and passing through the curve: this cone is a cubic cone, and every edge of it is a chord of the quartic. The equations of the chord through a and a variable point θ are

$$\frac{x-ay-\theta(y-az)}{\alpha-a\beta-\theta(\beta-a\gamma)}=\frac{y-az-\theta(z-au)}{\beta-a\gamma-\theta(\gamma-a\delta)}=\frac{z-au-\theta u}{\gamma-a\delta-\theta(a+\delta)},$$

the locus of this line for different values of θ is the required cone. Each of the fractions is equal to

$$\frac{\begin{vmatrix} x-ay, & y-az, & z-au \\ y-az, & z-au, & u \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}}{\begin{vmatrix} \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \\ y-az, & z-au, & u \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}},$$

and also to

$$\frac{-\theta\begin{vmatrix} y-az, & z-au, & u \\ x-ay, & y-az, & z-au \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \end{vmatrix}}{-\theta\begin{vmatrix} \beta-a\gamma, & \gamma-a\delta, & \delta+a \\ x-ay, & y-az, & z-au \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \end{vmatrix}};$$

and therefore the cubic cone having its vertex at a and passing through the quartic is

$$\frac{\begin{vmatrix} x-ay, & y-az, & z-au \\ y-az, & z-au, & u \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}}{\begin{vmatrix} y-az, & z-au, & \gamma-a\delta \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}}=\frac{\begin{vmatrix} x-ay, & y-az, & z-au \\ y-az, & z-au, & u \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \end{vmatrix}}{\begin{vmatrix} x-ay, & y-az, & z-au \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}}.$$

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And it is easy to prove that the line

$$\begin{vmatrix} y-az, & z-au, & u \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix} = 0, \quad = \begin{vmatrix} x-ay, & y-ax, & z-au \\ \alpha-a\beta, & \beta-a\gamma, & \gamma-a\delta \\ \beta-a\gamma, & \gamma-a\delta, & \delta+a \end{vmatrix}$$

is a double edge of the cubic cone.

The quartic is (§ 6) the partial intersection of this cubic cone with the quadric

$$\begin{vmatrix} y, & x+\beta u, & \alpha+\beta\delta \\ z, & y+\gamma u, & \beta+\gamma\delta \\ u, & z+\delta u, & \gamma+\delta^2 \end{vmatrix} = 0.$$

13. The equations of the chord through a and b are

$$\frac{x-y\theta+z\phi}{\alpha-\beta\theta+\gamma\phi} = \frac{y-z\theta+u\phi}{\beta-\gamma\theta+\delta\phi} = \frac{z-u\theta}{\gamma-\delta\theta-\phi},$$

where $\theta = a+b$, $\phi = ab$; they can also be taken in the form

$$\frac{x-\alpha-(y-\beta)\theta+(z-\gamma)\phi}{\alpha-\beta\theta+\gamma\phi} = \frac{y-\beta-(z-\gamma)\theta+(u-\delta)\phi}{\beta-\gamma\theta+\delta\phi}$$

$$= \frac{z-\gamma-(u-\delta)\theta+\phi}{\gamma-\delta\theta-\phi}.$$

No line can meet the quartic in more than three points, for otherwise a plane through the line and one point of the curve not lying upon the line would meet the quartic in more than four points. The preceding chord meets the curve in a and b: it may possibly meet it in a third point. Let it meet the curve in a point ξ; the values of ξ are given by

$$\frac{\xi^2(\xi^2-\xi\theta+\phi)}{\alpha-\beta\theta+\gamma\phi} = \frac{\xi(\xi^2-\xi\theta+\phi)}{\beta-\gamma\theta+\delta\phi} = \frac{\xi^2-\xi\theta+\phi}{\gamma-\delta\theta-\phi}.$$

The points a and b are given by

$$\xi^2-\xi\theta+\phi = 0;$$

if the chord can meet the curve in a third point ω, then

$$\frac{\omega^3}{\alpha-\beta\theta+\gamma\phi} = \frac{\omega}{\beta-\gamma\theta+\delta\phi} = \frac{1}{\gamma-\delta\theta-\phi},$$

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so that

$$(\alpha - \beta\theta + \gamma\phi)(\gamma - \delta\theta - \phi) = (\beta - \gamma\theta + \delta\phi)^2.$$

This is a quadratic equation in b: it is easy to see from general considerations or by actual calculation from the quadratic, using

$$\omega = \frac{\beta - \gamma\theta + \delta\phi}{\gamma - \delta\theta - \phi},$$

that its roots are b and ω. Hence, it follows that *through each point of the curve one, and only one, line can be drawn cutting it in other two points; if a be the given point, the triple secant through a cuts the curve again in ω and ω', where ω and ω' are the roots of the quadratic*

$$\begin{aligned}&\omega^2\{(\gamma - a\delta)^2 - (\beta - a\gamma)(\delta + a)\}\\ &+ \omega\{(\alpha - a\beta)(\delta + a) - (\beta - a\gamma)(\gamma - a\delta)\}\\ &+ (\beta - a\gamma)^2 - (\alpha - a\beta)(\gamma - a\delta) = 0.\end{aligned}$$

And the equations of the triple secant through a are

$$\frac{x - y\theta + z\phi}{\alpha - \beta\theta + \gamma\phi} = \frac{y - z\theta + u\phi}{\beta - \gamma\theta + \delta\phi} = \frac{z - u\theta}{\gamma - \delta\theta - \phi},$$

where $\theta = \omega + \omega'$, $\phi = \omega\omega'$.

14. In the same way it may be proved that *there are four tangents to the curve which meet the curve again in one other point.* If a denote the point where the line touches the curve, and ξ the point where it meets the curve again, then

$$\frac{\xi^3}{\alpha - 2a\beta + a^2\gamma} = \frac{\xi}{\beta - 2a\gamma + a^2\delta} = \frac{1}{\gamma - 2a\delta - a^2};$$

whence a satisfies the equation

$$(\alpha - 2a\beta + a^2\gamma)(\gamma - 2a\delta - a^2) = (\beta - 2a\gamma + a^2\delta)^2,$$

that is,

$$\begin{aligned}a^4(\gamma + \delta^2) - 2a^3(\beta + \gamma\delta) + a^2(\alpha - 2\beta\delta + 3\gamma^2)\\ + 2a(\alpha\delta - \beta\gamma) + \beta^2 - \alpha\gamma = 0,\end{aligned}$$

so that there are four such points.

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The condition that these four points should lie in one plane is found to be

$$\alpha(\gamma + \delta^2) - \beta^2 - 2\beta\gamma\delta + \gamma^3 = 0,$$

on substituting in the equation of § 7; hence the quartic is of the first species. Therefore *the points of contact of the four tangents to a quartic of the first species, which meet the curve in one other point, lie in a plane.*

15. Consider a family of planes through the chord ab: the intersection of any one of the planes with the curve is two points c and d, the chord joining which is

$$\frac{x - y\epsilon + z\eta}{\alpha - \beta\epsilon + \gamma\eta} = \frac{y - z\epsilon + u\eta}{\beta - \gamma\epsilon + \delta\eta} = \frac{z - u\epsilon}{\gamma - \delta\epsilon - \eta},$$

where $\epsilon = c + d$, $\eta = cd$, and

$$(\alpha - \beta\theta + \gamma\phi) - \epsilon(\beta - \gamma\theta + \delta\phi) + \eta(\gamma - \delta\theta - \phi) = 0.$$

When η is eliminated from the equations of the chord by the last relations, they take the form

$$\frac{U_1 - \epsilon V_1}{A_1 - \epsilon B_1} = \frac{U_2 - \epsilon V_2}{A_2 - \epsilon B_2} = \frac{U_3 - \epsilon V_3}{A_3 - \epsilon B_3},$$

where

$$U_1 = x - z\,\frac{\alpha - \beta\theta + \gamma\phi}{\gamma - \delta\theta - \phi},\quad A_1 = \alpha - \gamma\,\frac{\alpha - \beta\theta + \gamma\phi}{\gamma - \delta\theta - \phi},$$

$$V_1 = y - z\,\frac{\beta - \gamma\theta + \delta\phi}{\gamma - \delta\theta - \phi},\quad B_1 = \beta - \gamma\,\frac{\beta - \gamma\theta + \delta\phi}{\gamma - \delta\theta - \phi},$$

and so for the others. The result of eliminating ϵ between the two equations is found without difficulty to be

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ V_1, & V_2, & V_3 \\ A_1, & A_2, & A_3 \end{vmatrix} \begin{vmatrix} V_1, & V_2, & V_3 \\ A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \end{vmatrix} = \begin{vmatrix} U_1, & U_2, & U_3 \\ V_1, & V_2, & V_3 \\ B_1, & B_2, & B_3 \end{vmatrix} \begin{vmatrix} U_1, & U_2, & U_3 \\ A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \end{vmatrix},$$

so that* *the locus of the chord joining the two other points of intersection of the quartic by any plane through a given chord is a cubic surface on which the given chord is a double line.*

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* Cremona, l.c., p. 84.

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16. It has been proved that through any point in space (i) six osculating planes can be drawn to a twisted quartic of the second species, (ii) four double tangent planes can be drawn to it, (iii) three chords can be drawn, each meeting it in two points.

Take the point as the vertex of a quartic cone drawn through the curve; then* *a twisted quartic of the second species projects into a plane quartic that has three double points (the sections of the chords being double edges of the cone), four double tangents (the sections of the double tangent planes), and six inflexions (the sections of the six osculating planes that meet the curve elsewhere in a single point.)*

A corresponding result can be obtained by projecting *from a point on the curve; the projection is a cubic with one double point* (the intersection of the triple secant by the plane of projection) *and three inflexions.*

Further results, obtained by Cremona and others, can be established in a similar manner; sufficient results are given in this paper to indicate the application of an analytical method to the geometry of twisted quartics of the second species.

Application to curves of any degree.

17. The method evidently is capable of application to curves of any degree. A unicursal twisted curve of degree n can be represented by the equations

$$x = \theta^n + \sum_{s=0}^{n-4} \alpha_{n-s-4}\theta^s,$$

$$y = \theta^{n-1} + \sum_{s=0}^{n-4} \beta_{n-s-4}\theta^s,$$

$$z = \theta^{n-2} + \sum_{s=0}^{n-4} \gamma_{n-s-4}\theta^s,$$

$$u = \theta^{n-3} + \sum_{s=0}^{n-4} \delta_{n-s-4}\theta^s,$$

where the quantities $\alpha, \beta, \gamma, \delta$ are constants; and the n points in which the curve is cut by a plane

$$lx + my + nz + pu = 0$$

are determined by the roots of the equation

$$l\theta^n + m\theta^{n-1} + n\theta^{n-2} + p\theta^{n-3} + \sum_{s=0}^{n-4} (l\alpha_{n-s-4} + m\beta_{n-s-4} + n\gamma_{n-s-4} + p\delta_{n-s-4})\theta^s = 0.$$

* Cremona, l.c., p. 94.

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If this plane osculate the curve in a, there are $n-3$ other points in which it meets the curve: let these points be determined as the roots of

$$\xi^{n-3} - p_1\xi^{n-4} + p_2\xi^{n-5} - \ldots = 0.$$

Then taking $l=1$, we have

$$-m = 3a + p_1,$$

$$n = 3a^2 + 3ap_1 + p_2,$$

$$-p = a^3 + 3a^2p_1 + 3ap_2 + p_3,$$

$$\pm(\alpha_r + m\beta_r + n\gamma_r + p\delta_r) = a^3p_{r+1} + 3a^2p_{r+2} + 3ap_{r+3} + p_{r+4},$$

where in the last equation $r = 0, 1, 2, \ldots$; and p_{n-3}, p_{n-2} and p_n are zero quantities. Let

$$\alpha_r - 3a\beta_r + 3a^2\gamma_r - a^3\delta_r = A_r,$$

$$\beta_r - 3a\gamma_r + 3a^2\delta_r = B_r,$$

$$\gamma_r - 3a\delta_r = -C_r:$$

then

$$A_r = B_rp_1 + C_rp_2 + \delta_rp_3 \pm (a^3p_{r+1} + 3a^2p_{r+2} + 3ap_{r+3} + p_{r+4}),$$

which holds for $r = 0, 1, \ldots, n-4$ with the convention that p_{n-3}, p_{n-2}, p_n all vanish. When these $n-3$ equations are solved to determine the $n-3$ quantities $p_1, p_2, \ldots, p_{n-3}$, they lead to results of the form

$$\frac{p_1}{\Theta_1} = \frac{p_2}{\Theta_2} = \ldots = \frac{p_{n-3}}{\Theta_{n-3}} = \frac{1}{\Theta},$$

where Θ is of degree $3(n-3)$ in a and no one of the quantities $\Theta_1, \ldots, \Theta_{n-3}$ is of degree higher than $3(n-3)$.

The equation of the osculating plane at a is therefore of degree $3+3(n-3)$, that is, $3n-6$, in a when it is made integral in a; consequently *through any point in space* $3n-6$ *osculating planes can be drawn to an unicursal twisted curve of degree* n.

Again, if the osculating plane pass through a fourth consecutive point of the curve, so that the contact is triple and the plane stationary, a must satisfy the equation

$$\xi^{n-3} - p_1\xi^{n-4} + p_2\xi^{n-5} - \ldots = 0;$$

hence a is determined by

$$a^{n-4}\Theta - a^{n-4}\Theta_1 + \ldots = 0,$$

which is of degree $4(n-3)$; consequently *there are* $4n-12$

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points on an unicursal twisted curve of degree n at which the osculating plane is stationary.

It is evident that a cannot be a repeated root of the equation in ζ unless the appropriate condition between the parameters $\alpha, \beta, \gamma, \delta$ of the curve be satisfied; hence *there are no points on a general unicursal twisted curve of degree n at which the contact with the osculating plane is quadruple.*

18. Proceeding as in § 11 for the equations of the chord joining the two points determined by a and b, we can obtain them in the form

$$\left\| \begin{array}{llll}
x-\theta y+\phi z & , y-\theta z+\phi u & , z-\theta u & , \\
\alpha_0-\theta\beta_0+\phi\gamma_0 & , \beta_0-\theta\gamma_0+\phi\delta_0 & , \gamma_0-\theta\delta_0-\phi, & \delta_0+\theta, 1, 0, 0, 0, \ldots, 0, 0 \\
\alpha_1-\theta\beta_1+\phi\gamma_1 & , \beta_1-\theta\gamma_1+\phi\delta_1 & , \gamma_1-\theta\delta_1 & , \delta_1+\phi, \theta, 1, 0, 0, \ldots, 0, 0 \\
\alpha_2-\theta\beta_2+\phi\gamma_2 & , \beta_2-\theta\gamma_2+\phi\delta_2 & , \gamma_2-\theta\delta_2 & , \delta_2 , \phi, \theta, 1, 0, \ldots, 0, 0 \\
\alpha_3-\theta\beta_3-\phi\gamma_3 & , \beta_3-\theta\gamma_3+\phi\delta_3 & , \gamma_3-\theta\delta_3 & , \delta_3 , 0, \phi, \theta, 1, \ldots, 0, 0 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\alpha_{n-3}-\theta\beta_{n-3}+\phi\gamma_{n-3}, & \beta_{n-3}-\theta\gamma_{n-3}+\phi\delta_{n-3}, & \gamma_{n-3}-\theta\delta_{n-3}, & \delta_{n-3} , 0, 0, 0, 0, \ldots, \phi, \theta \\
\alpha_{n-4}-\theta\beta_{n-4}+\phi\gamma_{n-4}, & \beta_{n-4}-\theta\gamma_{n-4}+\phi\delta_{n-4}, & \gamma_{n-4}-\theta\delta_{n-4}, & \delta_{n-4} , 0, 0, 0, 0, \ldots, 0, \phi
\end{array} \right\| = 0.$$

From these equations it may be inferred that *the number of lines which can be drawn through an arbitrary point in space, so as to meet the curve in two points, is* 3^{n-3}; but the number may be less than this integer when special relations exist among the parametric constants of the twisted curve.

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PRODUCTS AND SERIES INVOLVING PRIME NUMBERS ONLY.

By J. W. L. GLAISHER.

M. Rogel's Results, §§ 1, 2.

§ 1. IN the *Messenger*, vol. xxiii. (pp. 169, 170) reference was made to some interesting theorems given by M. Rogel in the *Educational Times* Reprint (Vol. LV., p. 66).

The general theorem is that, n being any integral number, and μ being any real quantity intermediate to 0 and 1,

$$\Pi_{n=2}^{n=\infty} n^{\frac{\sin 2n\mu\pi}{n}} = \left\{\frac{(\sin\mu\pi)^{\frac{1}{2}}\Gamma(\mu)}{2^{\frac{1}{2}-\mu}\pi^{1-\mu}e^{(\frac{1}{2}-\mu)\gamma}}\right\}^{\pi},$$

where γ is Euler's constant.

By putting $\mu=\frac{1}{4}$ and $\mu=\frac{1}{8}$, M. Rogel obtained the results

$$\frac{5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\ldots(4n+1)^{\frac{1}{4n+1}}}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}\ldots(4n-1)^{\frac{1}{4n-1}}} = \left(\frac{2K_0}{\pi e^{\frac{1}{4}\gamma}}\right)^{\frac{1}{4}\pi},$$

$$\frac{4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\ldots(3n+1)^{\frac{1}{3n+1}}}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}\ldots(3n-1)^{\frac{1}{3n-1}}} = \left(\frac{2^{\frac{1}{2}}3^{\frac{1}{2}}K_1}{\pi e^{\frac{1}{4}\gamma}}\right)^{\frac{2\pi}{3\sqrt{3}}},$$

where K_0 and K_1 are the complete elliptic integrals corresponding to the moduli $\sin 45°$ and $\sin 15°$ respectively.

§ 2. In the same article M. Rogel has also expressed these last two theorems in a form in which only prime numbers occur, the resulting formulæ being

$$\Pi_{p=3}^{p=\infty} p^{\frac{1}{(-1)^{\frac{1}{2}(p-1)}p-1}} = \frac{4K_0^2}{\pi^2 e^{\gamma}},$$

$$\Pi_{p=2}^{p=\infty} p^{\frac{1}{ip-1}} = \frac{2^{\frac{3}{2}}.3K_1^2}{\pi^2 e^{\gamma}},$$

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where p is any prime, and $\epsilon = +1$ or -1 according as $p \equiv 1$ or $\equiv 2$, mod. 3.

The present paper relates to this second form of these and similar results, *i.e.* in which only prime numbers occur in the products.

Proofs of the formulæ, §§ 3–5.

§ 3. M. Rogel obtained the latter formulæ by decomposing into their prime factors the numbers which occur in the first form of the products (§ 1), and determining the exponent of any prime number p.

This process may be exhibited in a slightly different form as follows.

Taking the first product, viz.

$$3^{-\frac{1}{3}}5^{\frac{1}{5}}7^{-\frac{1}{7}}9^{\frac{1}{9}}11^{-\frac{1}{11}}13^{\frac{1}{13}}15^{-\frac{1}{15}}17^{\frac{1}{17}}19^{-\frac{1}{19}}21^{\frac{1}{21}}23^{-\frac{1}{23}}\ldots,$$

it is evident that, supposing every number (but not the exponents) decomposed into its prime factors, the exponent of 3

$$= -\tfrac{1}{3} + \tfrac{2}{3} - \tfrac{1}{15} + \tfrac{1}{21} - \tfrac{2}{27} + \tfrac{1}{33} - \tfrac{1}{39} + \&c.,$$

(in which the numerator is equal to the exponent of the highest power of 3 which occurs in the denominator)

$$= -\tfrac{1}{3}\{1 - \tfrac{2}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{2}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{2}{15} + \&c.\}$$

$$= -\tfrac{1}{3}(1+\tfrac{1}{3})^{-2}(1-\tfrac{1}{5})^{-1}(1+\tfrac{1}{7})^{-1}(1+\tfrac{1}{11})^{-1}\ldots$$

$$= -\tfrac{1}{3}(1+\tfrac{1}{3})^{-1}\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \&c.\} = -\frac{1}{3+1}\frac{\pi}{4}.$$

Similarly, the exponent of 5

$$= \tfrac{1}{5} - \tfrac{1}{15} + \tfrac{2}{25} - \tfrac{1}{35} + \tfrac{1}{45} - \&c.$$

$$= \tfrac{1}{5}\{1 - \tfrac{1}{3} + \tfrac{2}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \&c.\}$$

$$= \tfrac{1}{5}(1+\tfrac{1}{3})^{-1}(1-\tfrac{1}{5})^{-2}(1+\tfrac{1}{7})^{-1}(1+\tfrac{1}{11})^{-1}\ldots$$

$$= \tfrac{1}{5}(1-\tfrac{1}{5})^{-1}\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \&c.\} = \frac{1}{5-1}\frac{\pi}{4}.$$

This reasoning is clearly of general application, and the exponent of p is evidently equal to $\frac{1}{p-1}\frac{\pi}{4}$ or $\frac{1}{p+1}\frac{\pi}{4}$, according as p is of the form $4n+1$ or $4n+3$.

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Thus, the equation becomes

$$\left\{\frac{5^{\frac{1}{5-1}}13^{\frac{1}{13-1}}17^{\frac{1}{17-1}}29^{\frac{1}{29-1}}\ldots}{3^{\frac{1}{3+1}}7^{\frac{1}{7+1}}11^{\frac{1}{11+1}}19^{\frac{1}{19+1}}\ldots}\right\}^{\frac{1}{4\pi}} = \left(\frac{2K_0}{\pi e^{\frac{1}{12}}}\right)^{\frac{1}{2\pi}},$$

that is,

$$\frac{5^{\frac{1}{4}}13^{\frac{1}{12}}17^{\frac{1}{16}}29^{\frac{1}{28}}\ldots}{3^{\frac{1}{4}}7^{\frac{1}{8}}11^{\frac{1}{12}}19^{\frac{1}{20}}\ldots} = \frac{4K_0^2}{\pi^2 e^{\gamma}}.$$

§ 4. Treating in a similar manner the second formula, we see that the exponent of 2 in the product

$$2^{-\frac{1}{2}}4^{\frac{1}{4}}5^{-\frac{1}{6}}7^{\frac{1}{8}}8^{-\frac{1}{8}}10^{\frac{1}{10}}11^{-\frac{1}{12}}13^{\frac{1}{12}}14^{-\frac{1}{14}}16^{\frac{1}{16}}17^{-\frac{1}{17}}\ldots$$

is

$$\begin{aligned}
&-\tfrac{1}{2}+\tfrac{2}{4}-\tfrac{3}{8}+\tfrac{4}{10}-\tfrac{5}{14}+\tfrac{6}{16}-\tfrac{7}{20}+\tfrac{8}{22}-\ldots\\
&\quad=-\tfrac{1}{2}\{1-\tfrac{3}{2}+\tfrac{4}{4}-\tfrac{5}{6}+\tfrac{6}{7}-\tfrac{7}{8}+\tfrac{8}{10}-\tfrac{9}{11}+\ldots\}\\
&\quad=-\tfrac{1}{2}(1+\tfrac{1}{2})^{-2}(1+\tfrac{1}{3})^{-1}(1-\tfrac{1}{4})^{-1}(1+\tfrac{1}{11})^{-1}(1-\tfrac{1}{13})^{-1}\ldots\\
&\quad=-\tfrac{1}{2}(1+\tfrac{1}{2})^{-1}\{1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{10}-\&c.\}\\
&\quad=-\frac{1}{2+1}\frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

In the same way we see that the exponent of 5 is $-\frac{1}{5+1}\frac{\pi}{3\sqrt{3}}$, that the exponent of 7 is $\frac{1}{7-1}\frac{\pi}{3\sqrt{3}}$, and so on.

Thus, the equation becomes

$$\left\{\frac{7^{\frac{1}{7-1}}13^{\frac{1}{13-1}}19^{\frac{1}{19-1}}31^{\frac{1}{31-1}}\ldots}{2^{\frac{1}{2+1}}5^{\frac{1}{5+1}}11^{\frac{1}{11+1}}17^{\frac{1}{17+1}}\ldots}\right\}^{\frac{\pi}{8\sqrt{3}}} = \left(\frac{2^{\frac{1}{2}}3^{\frac{1}{4}}K_1}{\pi e^{\frac{1}{12}}}\right)^{\frac{2\pi}{8\sqrt{3}}},$$

that is,

$$\frac{7^{\frac{1}{8}}13^{\frac{1}{12}}19^{\frac{1}{18}}31^{\frac{1}{30}}\ldots}{2^{\frac{1}{2}}5^{\frac{1}{6}}11^{\frac{1}{12}}17^{\frac{1}{16}}\ldots} = \frac{2^{\frac{3}{2}}3K_1^2}{\pi^2 e^{\gamma}}.$$

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§ 5. In this product the numbers in the numerator are the primes of the form $3n+1$, and the numbers in the denominator are those of the form $3n+2$. With the exception of 2 all prime numbers are of the forms $6n+1$ or $6n+5$, so that we may conveniently express this result in the form

$$\frac{7^{\frac{1}{6}}13^{\frac{1}{12}}19^{\frac{1}{12}}31^{\frac{1}{30}}\ldots}{5^{\frac{1}{6}}11^{\frac{1}{12}}17^{\frac{1}{12}}23^{\frac{1}{12}}\ldots} = \frac{6K_1^2}{\pi^2 e^{\gamma}} = \frac{2K_2^2}{\pi^2 e^{\gamma}},$$

where the numerator and denominator contain respectively the primes of the forms $6n+1$ and $6n+5$, and K_2 is (as in the paper referred to in § 1), the complete elliptic integral to modulus $\sin 75°$.

Other applications of the method, §§ 6–14.

§ 6. The above method of expressing the products by means of prime numbers only may be applied also to other results which have been given in recent papers in the *Messenger*.

Thus, taking the quotient-product given on p. 42 of Vol. XXIV., we have

$$\frac{3^{\frac{1}{8}}9^{\frac{1}{8}}11^{\frac{1}{16}}17^{\frac{1}{16}}19^{\frac{1}{16}}\ldots}{5^{\frac{1}{8}}7^{\frac{1}{8}}13^{\frac{1}{16}}15^{\frac{1}{16}}21^{\frac{1}{16}}\ldots} = \left\{\frac{(\sin\frac{1}{8}\pi)\,\Gamma^2(\frac{1}{8})}{2^{\frac{1}{4}}\pi e^{\frac{1}{4}\gamma}\,\Gamma(\frac{1}{4})}\right\}^{\frac{\pi}{\sqrt{2}}}$$

$$= \left\{\frac{2^{\frac{1}{4}}(\sin\frac{1}{8}\pi)\,K_2}{\pi e^{\frac{1}{4}\gamma}}\right\}^{\frac{\pi}{\sqrt{2}}} = \left\{\frac{4(\sqrt{2}-1)^{\frac{1}{2}}K_2}{\pi e^{\frac{1}{4}\gamma}}\right\}^{\frac{\pi}{\sqrt{2}}},$$

where in the product the numbers in the numerator are of the forms $8n+1$ and $8n+3$, and those in the denominator are of the forms $8n+5$ and $8n+7$, and K_2 is the complete integral to modulus $\sqrt{2}-1$.

Proceeding as in § 3, it is evident that the exponent of 3

$$= \tfrac{1}{8} + \tfrac{2}{9} - \tfrac{1}{15} - \tfrac{1}{21} + \tfrac{2}{17} + \tfrac{1}{33} - \ldots,$$

$$= \tfrac{1}{8}(1 + \tfrac{2}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{2}{9} + \tfrac{1}{11} - \ldots),$$

$$= \tfrac{1}{8}(1 - \tfrac{1}{3})^{-1}(1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \ldots),$$

since the product of a number of the form $8n+1$ or $8n+3$ by a number of the form $8n+1$ or $8n+3$ is of one of these forms; but if such a number be multiplied by a number of the form $8n+5$ or $8n+7$, the product is of one of these latter forms.

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[illegible] product and series

[illegible]

$$= \frac{\quad}{\quad} = \frac{\quad}{\quad}.$$

[illegible]

$$= \quad \&c.,$$

$$= \quad \&c.,$$

$$= \frac{\quad}{\quad}$$

[illegible]

[illegible]

$$= \frac{\quad}{\quad}$$

$$= \frac{\quad}{\quad},$$

[illegible] the numerators are the numbers in the denominator are the primes [illegible] and [illegible] and those in the denominator are the primes of the forms $8n-5$ and $8n-3$.

[illegible] it was shown that

$$\frac{\quad}{\quad} = \frac{\quad}{\quad}.$$

[illegible] the numerator being primes of the form [illegible] and in the denominator of the form $8n+5$, and by [illegible] the formula obtained in the last section by this [illegible] we have

$$\frac{\quad}{\quad} = \frac{\quad}{\quad},$$

[illegible]

$$\frac{\quad}{\quad} = \frac{\quad}{\quad},$$

the numerators in the numerator being the primes of the form $8n+3$ and those in the denominator of the form $8n+5$.

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Expressed in terms of Gamma Functions, the value of this product is

$$\frac{(\sqrt{2}-1)^{\frac{1}{2}}}{2^{\frac{1}{4}}}\,\frac{\Gamma(\frac{1}{8})\,\Gamma^3(\frac{1}{8})}{\Gamma^3(\frac{1}{4})}.$$

§ 8. The result obtained by multiplying the formulæ in §§ 6 and 7 may also be noticed. We thus find

$$\left(\frac{17^{\frac{1}{16}}31^{\frac{1}{16}}47^{\frac{1}{16}}\ldots}{7^{\frac{1}{8}}23^{\frac{1}{16}}31^{\frac{1}{16}}\ldots}\right)^2 3^{\frac{1}{2}-\frac{1}{4}}5^{\frac{1}{4}-\frac{1}{6}}11^{\frac{1}{10}-\frac{1}{12}}\ldots$$

$$=\frac{64\,(\sqrt{2}-1)\,K_0^2K_2^2}{\pi^4e^{2\gamma}}=\frac{(\sqrt{2}-1)\,\Gamma^2(\frac{1}{4})\,\Gamma^4(\frac{1}{8})}{2^{\frac{3}{2}}\pi^6e^{2\gamma}},$$

or, extracting the square root,

$$\frac{17^{\frac{1}{16}}31^{\frac{1}{16}}47^{\frac{1}{16}}\ldots}{7^{\frac{1}{8}}23^{\frac{1}{16}}31^{\frac{1}{16}}\ldots}\,3^{\frac{1}{3^2-1}}5^{\frac{1}{5^2-1}}11^{\frac{1}{11^2-1}}13^{\frac{1}{13^2-1}}\ldots$$

$$=\frac{8\,(\sqrt{2}-1)^{\frac{1}{2}}K_0K_2}{\pi^3e^{\gamma}}=\frac{(\sqrt{2}-1)^{\frac{1}{2}}\,\Gamma(\frac{1}{4})\,\Gamma^2(\frac{1}{8})}{2^{\frac{3}{4}}\pi^{\frac{9}{4}}e^{\gamma}}.$$

In the quotient on the left-hand side the numbers in the numerator are the primes of the form $8n+1$, and those in the denominator are the primes of the form $8n+7$. The product which follows the quotient contains the primes of the forms $8n+3$ and $8n+7$.

§ 9. The products derived from the series considered in a paper in vol. xxiv. of the *Messenger*,* in which the exponents are inverse powers of the numbers, admit of being similarly expressed by prime numbers only.

Thus, from p. 3, we have

$$2^{\frac{1}{2}}3^{\frac{1}{3}}4^{\frac{1}{4}}5^{\frac{1}{5}}6^{\frac{1}{6}}\ldots=\left(\frac{A^{12}}{2\pi e^{\gamma}}\right)^{\frac{1}{4}\pi^2},$$

where A is the constant 1·282427130, which forms the subject of that paper.

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* 'On the constant which occurs in the formula for $1^1.2^2.3^3\ldots n^n$,' pp. 1–16.

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In this product, the exponent of 2

$$= \tfrac{1}{4} + \tfrac{2}{16} + \tfrac{1}{36} + \tfrac{3}{64} + \tfrac{1}{100} + \&c.$$

$$= \tfrac{1}{2}\left(1 + \tfrac{2}{4} + \tfrac{1}{9} + \tfrac{3}{16} + \tfrac{1}{25} + \&c.\right)$$

$$= \tfrac{1}{4}\left(1 - \frac{1}{2^2}\right)^{-1}\left(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \&c.\right)$$

$$= \frac{1}{2^2-1}\cdot\frac{\pi^2}{6}.$$

Similarly, the exponent of 3 is $\frac{1}{3^2-1}\cdot\frac{\pi^2}{6}$, that of 5 is $\frac{1}{5^2-1}\cdot\frac{\pi^2}{6}$, and so on.

Thus, by extracting the $\left(\frac{6}{\pi^2}\right)^{\text{th}}$ root of each side, we deduce the result

$$2^{\frac{1}{2^2-1}}3^{\frac{1}{3^2-1}}5^{\frac{1}{5^2-1}}7^{\frac{1}{7^2-1}}\ldots = \frac{A^{12}}{2\pi e^{\gamma}},$$

in which all the prime numbers occur on the left-hand side, each prime p being raised to the power $\frac{1}{p^2-1}$.

§ 10. By taking the logarithm of this result, we have

$$\frac{\log 2}{2^2-1} + \frac{\log 3}{3^2-1} + \frac{\log 5}{5^2-1} + \frac{\log 7}{7^2-1} + \&c.$$

$$= 12\log A - \log 2 - \log\pi - \gamma.$$

§ 11. If we apply the process of § 9 to the formula

$$3^{\frac{1}{8}}5^{\frac{1}{24}}7^{\frac{1}{48}}9^{\frac{1}{64}}\ldots = \left(\frac{A^3}{2^{\frac{1}{4}}\pi^{\frac{1}{4}}e^{\frac{1}{4}\gamma}}\right)^{\frac{1}{4}\pi^2}$$

(*Messenger*, Vol. XXIII., p. 173), in which only uneven numbers occur, we find

$$3^{\frac{1}{3^2-1}}5^{\frac{1}{5^2-1}}7^{\frac{1}{7^2-1}}11^{\frac{1}{11^2-1}}\ldots = \frac{A^{12}}{2^{\frac{1}{4}}\pi e^{\gamma}},$$

which is equivalent to the result obtained in § 9.

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§ 12. It was shown on p. 11 of Vol. XXIV. of the *Messenger* that

$$\frac{5^{\frac{1}{5^2}}9^{\frac{1}{9^2}}13^{\frac{1}{13^2}}\ldots}{3^{\frac{1}{3^2}}7^{\frac{1}{7^2}}11^{\frac{1}{11^2}}\ldots} = \left\{\frac{A}{2^{\frac{1}{2^2}}\pi^{\frac{1}{2^2}}}e^{-\frac{3}{2^2}-\frac{1}{2^2}\gamma+\frac{1}{2^2}P}\right\}^{\pi^2},$$

where $$P = \frac{S_3}{3.4\,5}\frac{1}{4^3} + \frac{S_5}{5.6.7}\frac{1}{4^5} + \frac{S_7}{7.8.9}\frac{1}{4^7} + \&c.,$$

and $$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \frac{1}{5^r} + \&c.$$

In the quotient on the left-hand side, the exponent of 3

$$= -\frac{1}{3^3} + \frac{2}{9^3} - \frac{1}{15^3} + \frac{1}{21^3} - \frac{3}{27^3} + \&c.$$

$$= -\frac{1}{3^3}\left(1+\frac{1}{3^3}\right)^{-1}\left\{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.\right\}$$

$$= -\frac{1}{3^3+1}\cdot\frac{\pi^3}{32}.$$

Similarly the exponent of 5 is $\frac{1}{5^3-1}\cdot\frac{\pi^3}{32}$, that of 7 is $\frac{1}{7^3+1}\cdot\frac{\pi^3}{32}$, and so on.

Thus the formula may be transformed into

$$\frac{5^{\frac{1}{5^3-1}}13^{\frac{1}{13^3-1}}17^{\frac{1}{17^3-1}}\ldots}{3^{\frac{1}{3^3+1}}7^{\frac{1}{7^3+1}}11^{\frac{1}{11^3+1}}\ldots} = \frac{A^{32}}{32\pi}e^{-8-\frac{8}{3}\gamma+8P},$$

where P is as above. On the left-hand side the numbers in the numerator are the primes of the form $4n+1$ and those in the denominator the primes of the form $4n+3$.

§ 13. Since

$$1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \&c. = \frac{4\pi^3}{81\sqrt{3}},$$

we may similarly transform the formula given on p. 12,* viz.,

* On p. 12 the exponent on the right-hand side is printed $\frac{8\pi}{9\sqrt{3}}$ instead of $\frac{8\pi^3}{9\sqrt{3}}$.

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$$\frac{4^{\frac{1}{4^3}}7^{\frac{1}{7^3}}10^{\frac{1}{10^3}}\ldots}{2^{\frac{1}{2^3}}5^{\frac{1}{5^3}}8^{\frac{1}{8^3}}\ldots}=\left\{\frac{A^8}{2^{\frac{1}{12}}3^{\frac{1}{4}}\pi^{\frac{1}{12}}}e^{-\frac{1}{8}+Q}\right\}^{\frac{8\pi^3}{9\sqrt{3}}},$$

where $$Q=\frac{S_3}{3.4.5}\frac{1}{3^3}+\frac{S_5}{5.6.7}\frac{1}{3^5}+\frac{S_7}{7.8.9}\frac{1}{3^7}+\&c.,$$

into the following

$$\frac{7^{\frac{1}{7^3-1}}13^{\frac{1}{13^3-1}}19^{\frac{1}{19^3-1}}\ldots}{2^{\frac{1}{2^3+1}}5^{\frac{1}{5^3+1}}11^{\frac{1}{11^3+1}}\ldots}=\frac{A^{24}}{2.3^{\frac{3}{2}}.\pi}e^{-\frac{3}{4}+18Q},$$

where Q is as above. The numbers in the numerator are the primes of the form $3n+1$ and those in the denominator are the primes of the form $3n+2$. If we multiply throughout by $2^{\frac{3}{2}}$ the numerator and denominator contain respectively primes of the forms $6n+1$ and $6n+5$.

§ 14. From the formula at the top of p. 15* of the same volume it follows that

$$2^{\frac{1}{2^2}}3^{\frac{1}{3^4}}4^{\frac{1}{4^4}}5^{\frac{1}{5^4}}\ldots=\left\{\frac{A}{2^{\frac{3.7}{2^6}}\pi^{\frac{1}{2^3}}}e^{-\frac{1}{12}+R}\right\}^{\frac{8}{15}\pi^4},$$

where $$R=\frac{S_3}{3.4.5.6}\frac{1}{2^3}+\frac{S_5}{5.6.7.8}\frac{1}{2^5}+\frac{S_7}{7.8.9.10}\frac{1}{2^7}+\&c.$$

Transforming the product as in § 9, we find

$$2^{\frac{1}{2^4-1}}3^{\frac{1}{3^4-1}}5^{\frac{1}{5^4-1}}7^{\frac{1}{7^4-1}}\ldots=\frac{A^{48}}{2^{\frac{13}{8}}\pi}e^{-\frac{3}{2}+48R},$$

where R is as above, and the product on the left-hand side contains only the prime numbers.

Multiplying throughout by $2^{-\frac{1}{12}}$, this formula may be written

$$3^{\frac{1}{3^4-1}}5^{\frac{1}{5^4-1}}7^{\frac{1}{7^4-1}}11^{\frac{1}{11^4-1}}\ldots=\frac{A^{48}}{32\pi}e^{-\frac{3}{2}+48R}.$$

The result in this form may be deduced directly from the first formula in § 26 of the paper, (p. 14).

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* In the second line on p. 15 the factor π^4 has been omitted.

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The series $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3}$ + &c., §§ 15–17.

§ 15. The same principles enable us to express the general theorem (involving sines) by means of prime numbers only. In order to simplify the investigation it is convenient to consider first the value of the series

$$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \&c.$$

Denoting this series by $\phi(x)$, and using the notation $a(x)b$ to indicate that x is to be intermediate in value to a and b, then we know that

$$\phi(x) = \frac{\pi - x}{2}, \quad 0(x)\pi.$$

§ 16. Let $x = \pi + \xi$, where ξ is intermediate to 0 and π, then

$$\phi(x) = -\sin\xi + \frac{\sin 2\xi}{2} - \frac{\sin 3\xi}{3} + \frac{\sin 4\xi}{4} - \&c.$$

$$= -\frac{\xi}{2} = \frac{\pi - x}{2}.$$

Thus $\quad \phi(x) = \frac{\pi - x}{2}, \quad \pi(x)2\pi.$

Let $x = 2\pi + \xi$, ξ being as before, then

$$\phi(x) = \sin\xi + \frac{\sin 2\xi}{2} + \frac{\sin 3\xi}{3} + \frac{\sin 4\xi}{4} + \&c.$$

$$= \frac{\pi - \xi}{2} = \frac{3\pi - x}{2}.$$

Thus $\quad \phi(x) = \frac{3\pi - x}{2}, \quad 2\pi(x)3\pi.$

Similarly by putting $x = 3\pi + \xi$, $4\pi + \xi$, &c., ξ being as before, we find

$$\phi(x) = \frac{3\pi - x}{2}, \quad 3\pi(x)4\pi,$$

$$\phi(x) = \frac{5\pi - x}{2}, \quad 4\pi(x)5\pi,$$

and so on.

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§ 17. The general formula therefore is

$$\phi(x) = \frac{(2n+1)\pi - x}{2}, \quad 2n\pi\,(x)\,(2n+2)\,\pi,$$

and therefore putting πx for x, we find

$$\sin\pi x + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \frac{\sin 4\pi x}{4} + \&c. = \tfrac{1}{2}\pi\,[x],$$

where $[x]$ denotes $n - x$, n being the *nearest* uneven integer to x. If x be an integer, $[x]$ is supposed to be zero.

The general theorem, §§ 18–20.

§ 18. Putting $\frac{1}{2}a$ for μ in M. Rogel's general theorem (§ 1), it may be written

$$\Pi_{n=2}^{n=\infty} n^{\frac{\sin na\pi}{n}} = \left\{\frac{\sin\frac{1}{2}a\pi\;\Gamma^2(\frac{1}{2}a)}{2^{1-a}\pi^{1-a}e^{(1-a)\gamma}}\right\}^{\frac{1}{\pi}}, \quad 0\,(a)\,1.$$

Writing the product on the left-hand side at full length, viz.

$$2^{\frac{\sin 2\pi a}{2}}\,3^{\frac{\sin 3\pi a}{3}}\,4^{\frac{\sin 4\pi a}{4}}\,5^{\frac{\sin 5\pi a}{5}}\,6^{\frac{\sin 6\pi a}{6}}\,7^{\frac{\sin 7\pi a}{7}}\,8^{\frac{\sin 8\pi a}{8}}\ldots,$$

we see that when the numbers are decomposed into their prime factors, the exponent of 2

$$= \frac{\sin 2\pi a}{2} + 2\,\frac{\sin 4\pi a}{4} + \frac{\sin 6\pi a}{6} + 3\,\frac{\sin 8\pi a}{8} + \frac{\sin 10\pi a}{10} + \&c.,$$

that is,

$$\begin{aligned}
&\frac{\sin 2\pi a}{2} + \frac{\sin 4\pi a}{4} + \frac{\sin 6\pi a}{6} + \frac{\sin 8\pi a}{8} + \&c.\\
&+ \frac{\sin 4\pi a}{4} + \frac{\sin 8\pi a}{8} + \frac{\sin 12\pi a}{12} + \frac{\sin 16\pi a}{16} + \&c.\\
&+ \frac{\sin 8\pi a}{8} + \frac{\sin 16\pi a}{16} + \frac{\sin 24\pi a}{24} + \frac{\sin 32\pi a}{32} + \&c.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

which by § 17,

$$= \tfrac{1}{2}\pi\left\{\tfrac{1}{2}[2a] + \tfrac{1}{4}[4a] + \tfrac{1}{8}[8a] + \tfrac{1}{16}[16a] + \&c.\right\}.$$

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Similarly, the exponent of 3

$$= \frac{\sin 3\pi a}{3} + \frac{\sin 6\pi a}{6} + 2\frac{\sin 9\pi a}{9} + \frac{\sin 12\pi a}{12} + \&c.$$

$$= \tfrac{1}{2}\pi\{\tfrac{1}{3}[3a] + \tfrac{1}{6}[9a] + \tfrac{1}{21}[27a] + \&c.\},$$

the exponent of 5

$$= \tfrac{1}{2}\pi\{\tfrac{1}{5}[5a] + \tfrac{1}{25}[25a] + \tfrac{1}{125}[125a] + \&c.\},$$

and so on.

§ 19. Thus, extracting the $\left(\frac{2}{\pi}\right)^{\text{th}}$ root of each side of the equation, we find

$$2^{\frac{1}{4}[2a]+\frac{1}{4}[4a]+\frac{1}{8}[8a]+\dots}\, 3^{\frac{1}{3}[3a]+\frac{1}{6}[9a]+\dots}\, 5^{\frac{1}{5}[5a]+\frac{1}{25}[25a]+\dots}$$

$$= \frac{\sin \frac{1}{2}a\pi\ \Gamma^2(\frac{1}{2}a)}{(2e^\gamma)^{1-a}\pi^{\frac{1}{2}-a}} = \frac{\Gamma(\frac{1}{2}a)}{(2\pi e^\gamma)^{\frac{1}{2}-a}\Gamma(1-\frac{1}{2}a)}, \quad 0\,(a)\,1.$$

In this formula, as in § 17, $[x]$ denotes $x-n$, n being the nearest uneven integer to x, unless x is an integer, in which case $[x]=0$. Only prime numbers occur in the product on the left-hand side.

§ 20. It will be observed that the exponent $\frac{1}{2}\pi$ disappears from the general formula just as the other π-exponents disappeared in the particular cases considered in §§ 3, 4.

This seems to indicate that the result just obtained is the more fundamental form of the theorem. It is interesting also to notice how the sines which occur in the exponents in the original form of the theorem are replaced by purely numerical quantities in the second form.

The results given by the theorem for special values of a will now be considered.

The case $a=\frac{1}{2}$, § 21.

§ 21. When $a=\frac{1}{2}$, the exponent of $2=\frac{1}{2}[2a]+\frac{1}{4}[4a]+\&c.=0$, for $2a$, $4a$, &c. are integers, and therefore $[2a]=0$, $[4a]=0$, &c.

Since

$$[(4n+1)\,a] = [2n+\tfrac{1}{2}] = \tfrac{1}{2},$$

and

$$[(4n+3)\,a] = [2n+\tfrac{3}{2}] = -\tfrac{1}{2},$$

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it is evident that, if p be of the form $4n+1$, the exponent

$$\frac{1}{p}[pa]+\frac{1}{p^2}[p^2a]+\frac{1}{p^3}[p^3a]+\&c.$$

$$=\frac{1}{p}\cdot\frac{1}{2}+\frac{1}{p^2}\cdot\frac{1}{2}+\frac{1}{p^3}\cdot\frac{1}{2}+\&c.=\frac{1}{p-1}\cdot\frac{1}{2},$$

and, that if p be of the form $4n+3$, this exponent

$$=\frac{1}{p}\cdot-\frac{1}{2}+\frac{1}{p^2}\cdot\frac{1}{2}+\frac{1}{p^3}\cdot-\frac{1}{2}+\&c.=-\frac{1}{p+1}\cdot\frac{1}{2}.$$

Thus the product

$$=3^{-\frac{1}{2}\cdot\frac{1}{3+1}}5^{\frac{1}{2}\cdot\frac{1}{5-1}}7^{-\frac{1}{2}\cdot\frac{1}{7+1}}11^{-\frac{1}{2}\cdot\frac{1}{11+1}}\ldots,$$

and the right-hand side of the equation

$$=\frac{2^{-\frac{1}{2}}\Gamma^2(\frac{1}{4})}{(2e^\gamma)^{\frac{1}{2}}\pi^{\frac{1}{2}}}=\frac{2K_0}{\pi e^{\frac{1}{2}\gamma}};$$

whence, squaring,

$$3^{-\frac{1}{4}}5^{\frac{1}{4}}7^{-\frac{1}{8}}11^{-\frac{1}{12}}13^{\frac{1}{12}}\ldots=\frac{4K_0^2}{\pi^2e^\gamma},$$

which is M. Rogel's first result (§§ 2 and 3).

The case $a=\frac{2}{3}$, § 22.

§ 22. When $a=\frac{2}{3}$, the exponent of 3, viz.

$\frac{1}{3}[3a]+\frac{1}{9}[9a]+\&c.=0$, since $[3a]=0$, $[9a]=0$, &c.

Since

$$[(3n+1)a]=[2n+\tfrac{2}{3}]=\tfrac{1}{3},$$

and

$$[(3n+2)a]=[2n+\tfrac{4}{3}]=-\tfrac{1}{3},$$

we see that, if p be of the form $3n+1$, the exponent

$$\frac{1}{p}[pa]+\frac{1}{p^2}[p^2a]+\frac{1}{p^3}[p^3a]+\&c.$$

$$=\frac{1}{p}\cdot\frac{1}{3}+\frac{1}{p^2}\cdot\frac{1}{3}+\frac{1}{p^3}\cdot\frac{1}{3}+\&c.=\frac{1}{p-1}\cdot\frac{1}{3},$$

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and, if p be of the form $3n+2$, this exponent

$$= \frac{1}{p} \cdot -\frac{1}{3} + \frac{1}{p^3} \cdot \frac{1}{3} + \frac{1}{p^5} \cdot -\frac{1}{3} + \&c. = -\frac{1}{p+1}\frac{1}{3}.$$

Thus the product

$$= 2^{-\frac{1}{8}\frac{1}{2+1}5^{-\frac{1}{8}\frac{1}{5+1}7^{\frac{1}{8}\frac{1}{7-1}}11^{-\frac{1}{8}\frac{1}{11+1}13^{\frac{1}{8}\frac{1}{13-1}}}}} \ldots .$$

The right-hand side

$$= \frac{\frac{1}{2}\sqrt{3}\,\Gamma^3(\frac{1}{3})}{(2e^\gamma)^{\frac{1}{3}}\pi^{\frac{1}{2}}} = \frac{2^{\frac{2}{3}}3^{\frac{1}{2}}K_1^{\frac{2}{3}}}{e^{\frac{1}{3}\gamma}\pi^{\frac{1}{2}}},$$

whence, cubing,

$$2^{-\frac{1}{8}}5^{-\frac{1}{6}\frac{1}{7}}6^{11^{-\frac{1}{12}}13^{\frac{1}{12}}} \ldots = \frac{2^{\frac{2}{3}}.3.K_1^2}{\pi^2 e^\gamma},$$

which is M. Rogel's second result (§§ 2 and 4).

The case $a = \frac{1}{3}$, § 23.

§ 23. When $a = \frac{1}{3}$, the exponent of 2

$$= \tfrac{1}{2}[\tfrac{2}{3}] + \tfrac{1}{4}[\tfrac{1}{3}] + \tfrac{1}{8}[\tfrac{2}{3}] + \&c.$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{1}{4}\cdot -\tfrac{1}{3} + \tfrac{1}{8}\cdot\tfrac{1}{3} + \&c. = \tfrac{1}{3}\frac{1}{2+1},$$

and we have

$$[(6n+1)a] = [2n + \tfrac{1}{3}] = \tfrac{2}{3},$$

$$[(6n+5)a] = [2n + \tfrac{5}{3}] = -\tfrac{2}{3}.$$

Thus the formula becomes

$$2^{\frac{1}{3}\frac{1}{2+1}5^{-\frac{2}{3}\frac{1}{5+1}7^{\frac{2}{3}\frac{1}{7-1}}11^{-\frac{2}{3}\frac{1}{11+1}13^{\frac{2}{3}\frac{1}{13-1}}}}} \ldots$$

$$= \frac{\frac{1}{2}\,\Gamma^3(\frac{1}{3})}{(2e^\gamma)^{\frac{1}{3}}\pi^{\frac{1}{2}}} = \frac{2^{\frac{1}{3}}3^{\frac{1}{2}}K_1^{\frac{1}{3}}}{e^{\frac{1}{3}\gamma}\pi^{\frac{1}{2}}},$$

whence, raising both sides to the power $\frac{3}{2}$,

$$2^{\frac{1}{6}}5^{-\frac{1}{6}\frac{1}{7}}6^{11^{-\frac{1}{12}}13^{\frac{1}{12}}} \ldots = \frac{2^{\frac{1}{2}}.3.K_1^2}{\pi^2 e^\gamma},$$

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which is equivalent to the result obtained in the preceding section.

The case $a = \frac{1}{4}$, § 24.

§ 24. When $a = \frac{1}{4}$, the exponent of 2

$$= \tfrac{1}{2}[\tfrac{1}{2}] + \tfrac{1}{4}[1] + \tfrac{1}{8}[2] + \&c. = \tfrac{1}{4},$$

and we have

$$[(8n+1)a] = [2n + \tfrac{1}{4}] = \tfrac{3}{4},$$
$$[(8n+3)a] = [2n + \tfrac{3}{4}] = \tfrac{1}{4},$$
$$[(8n+5)a] = [2n + \tfrac{5}{4}] = -\tfrac{1}{4},$$
$$[(8n+7)a] = [2n + \tfrac{7}{4}] = -\tfrac{3}{4}.$$

Thus, if p is of the form $8n+1$, the exponent of p

$$= \frac{1}{p}\frac{3}{4} + \frac{1}{p^2}\frac{3}{4} + \frac{1}{p^3}\frac{3}{4} + \&c. = \frac{3}{4}\frac{1}{p-1};$$

if p is of the form $8n+3$, its exponent

$$= \frac{1}{p}\frac{1}{4} + \frac{1}{p^2}\frac{3}{4} + \frac{1}{p^3}\frac{1}{4} + \frac{1}{p^4}\frac{3}{4} + \&c.$$
$$= \frac{1}{4}\frac{p}{p^2-1} + \frac{3}{4}\frac{1}{p^2-1} = \frac{1}{4}\frac{p+3}{p^2-1}.$$

Similarly, if p is of the form $8n+5$, its exponent

$$= -\frac{1}{p}\frac{1}{4} + \frac{1}{p^2}\frac{3}{4} - \&c. = -\frac{1}{4}\frac{p-3}{p^2-1},$$

and, if p is of the form $8n+7$, it

$$= -\frac{1}{p}\frac{3}{4} + \frac{1}{p^2}\frac{3}{4} - \&c. = -\frac{3}{4}\frac{1}{p+1}.$$

The theorem therefore becomes

$$2^{\frac{1}{4}}3^{\frac{1}{4}}\frac{3+3}{3^2-1}5^{-\frac{1}{4}}\frac{5-3}{5^2-1}7^{-\frac{3}{4}}\frac{1}{7+1}11^{\frac{1}{4}}\frac{11+3}{11^2-1}\ldots = \frac{\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{(2e^\gamma)^{\frac{1}{4}}\pi^{\frac{7}{4}}},$$

or, raising both sides to the fourth power,

$$3^{\frac{3+3}{3^2-1}}5^{-\frac{5-3}{5^2-1}}7^{-\frac{3}{7+1}}11^{\frac{11+3}{11^2-1}}13^{-\frac{13-3}{13^2-1}}\ldots$$
$$= \frac{\sin^4\frac{1}{8}\pi\,\Gamma^8(\frac{1}{8})}{2^4e^{3\gamma}\pi^7} = \frac{2^{10}(\sqrt{2}-1)^3K_0^3K_3^4}{\pi^6e^{3\gamma}}.$$

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In this product the exponent of any prime p is

$$\frac{3}{p-1}, \quad \frac{p+3}{p^2-1}, \quad -\frac{p-3}{p^3-1}, \quad -\frac{3}{p+1},$$

according as p is of the form $8n+1$, $8n+3$, $8n+5$, or $8n+7$. The quantity K_ε was defined in § 6.

The case $a = \frac{3}{4}$, § 25.

§ 25. When $a = \frac{3}{4}$, the exponent of 2

$$= \tfrac{1}{2}[\tfrac{3}{2}] + \tfrac{1}{4}[3] + \&c. = -\tfrac{1}{4},$$

and we have

$$[(8n+1)a] = [2n+\tfrac{3}{4}] = \tfrac{1}{4},$$
$$[(8n+3)a] = [2n+\tfrac{9}{4}] = \tfrac{2}{4},$$
$$[(8n+5)a] = [2n+\tfrac{15}{4}] = -\tfrac{2}{4},$$
$$[(8n+7)a] = [2n+\tfrac{21}{4}] = -\tfrac{1}{4}.$$

Thus:—

if p is of the form $8n+1$, its exponent $= \frac{1}{4}\frac{1}{p-1}$,

" $8n+3$, " $= \frac{1}{4}\frac{3p+1}{p^2-1}$,

" $8n+5$, " $= -\frac{1}{4}\frac{3p-1}{p^2-1}$,

" $8n+7$, " $= -\frac{1}{4}\frac{1}{p+1}$.

Raising both sides of the equation to the fourth power, the theorem becomes

$$3^{\frac{9+1}{3^2-1}} 5^{-\frac{15-1}{5^2-1}} 7^{-\frac{1}{7+1}} 11^{\frac{33+1}{11^2-1}} 13^{-\frac{39-1}{13^2-1}} \ldots$$
$$= \frac{\sin^4 \frac{3}{8}\pi\, \Gamma^8(\frac{3}{8})}{e^7\pi^6} = \frac{2^6(\sqrt{2}-1)^2 K_\varepsilon^{-2} K_\varepsilon^4}{\pi^2 e^7}.$$

In the product, the exponent of any prime p is

$$\frac{1}{p-1}, \quad \frac{3p+1}{p^2-1}, \quad -\frac{3p-1}{p^2-1}, \quad -\frac{1}{p+1},$$

according as p is of the form $8n+1$, $8n+3$, $8n+5$, or $8n+7$.

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The case $a=\frac{1}{8}$, § 26.

§ 26. When $a=\frac{1}{8}$,

$$[(12n+2)a] = [2n+\tfrac{1}{4}] = \tfrac{3}{4},$$
$$[(12n+4)a] = [2n+\tfrac{2}{4}] = \tfrac{1}{4},$$
$$[(12n+8)a] = [2n+\tfrac{3}{4}] = -\tfrac{1}{4},$$
$$[(12n+16)a] = [(12n+4)a] = \tfrac{1}{4},$$

and so on. Therefore the exponent of 2

$$= \tfrac{1}{2}[2a] + \tfrac{1}{4}[4a] + \tfrac{1}{8}[8a] + \&c.$$
$$= \tfrac{1}{2}\tfrac{3}{4} + (\tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - 3\tfrac{1}{32} + \&c.)\tfrac{1}{8} = \tfrac{7}{16}.$$

Also

$$[(12n+3)a] = [(2n+\tfrac{1}{2}] = \tfrac{1}{2},$$
$$[(12n+9)a] = [(2n+\tfrac{3}{2}] = -\tfrac{1}{2},$$
$$[(12n+27)a] = [12n+3)a] = \tfrac{1}{2},$$

and so on. Therefore the exponent of 3

$$= (\tfrac{1}{3} - \tfrac{1}{9} + \tfrac{1}{27} - \tfrac{1}{81} + \&c.)\tfrac{1}{2} = \tfrac{1}{8}.$$

We also have

$$[(12n+1)a] = [2n+\tfrac{1}{6}] = \tfrac{5}{6},$$
$$[(12n+5)a] = [2n+\tfrac{5}{6}] = \tfrac{1}{6},$$
$$[(12n+7)a] = [2n+\tfrac{7}{6}] = -\tfrac{1}{6},$$
$$[(12n+11)a] = [2n+\tfrac{11}{6}] = -\tfrac{5}{6},$$

so that:—

if p be a prime of the form $12n+1$, its exponent $= \frac{5}{6}\frac{1}{p-1}$,

" " $12n+5$, " $= \frac{1}{6}\frac{p+5}{p^2-1}$,

" " $12n+7$, " $= -\frac{1}{6}\frac{p-5}{p^2-1}$,

" " $12n+11$, " $= -\frac{5}{6}\frac{1}{p+1}$.

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The theorem therefore gives

$$2^{\frac{7}{18}}3^{\frac{1}{8}}5^{\frac{1}{6}}\,\overline{5^{\frac{5+5}{5^2-1}}7}^{-\frac{1}{6}}\,\overline{7^{\frac{7-5}{7^2-1}}11}^{-\frac{5}{6}}\,\overline{11^{\frac{1}{11+1}}13}^{\frac{5}{6}}\,\overline{13^{\frac{1}{13-1}}}\ldots$$

$$= \frac{\sin \frac{1}{12}\pi\, \Gamma^3(\frac{1}{12})}{(2e^7)^{\frac{3}{4}}\pi^{\frac{1}{4}}};$$

whence, raising each side to the sixth power,

$$5^{\frac{5+5}{5^2-1}}\,\overline{7^{\frac{7-5}{7^2-1}}11}^{-\frac{5}{11+1}}\,\overline{13^{\frac{5}{13-1}}17^{\frac{17+5}{17^2-1}}}\ldots$$

$$= \frac{\sin^6 \frac{1}{12}\pi\, \Gamma^{12}(\frac{1}{12})}{2^{\frac{3}{2}}3^{\frac{3}{2}}e^{5\cdot7}\pi^{11}} = \frac{2^3 3^{\frac{1}{2}} K_0^6 K_1^4}{\pi^{10} e^{5\cdot7}}.$$

In the product, the exponent of any prime p is

$$\frac{5}{p-1},\quad \frac{p+5}{p^2-1},\quad -\frac{p-5}{p^2-1},\quad -\frac{5}{p+1},$$

according as p is of the form $12n+1$, $12n+5$, $12n+7$, or $12n+11$.

The case $a = \frac{5}{6}$, §§ 27, 28.

§ 27. When $a = \frac{5}{6}$,

$$[(12n+2)a] = [10n+\tfrac{5}{3}] = -\tfrac{2}{3},$$
$$[(12n+4)a] = [10n+\tfrac{10}{3}] = -\tfrac{1}{3},$$
$$[(12n+8)a] = [10n+\tfrac{20}{3}] = \tfrac{1}{3},$$
$$[(12n+16)a] = [(12n+4)a] = -\tfrac{1}{3},$$

and so on. Therefore the exponent of 2

$$= \tfrac{1}{2}.-\tfrac{2}{3} + (\tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32} + \&c.).-\tfrac{1}{3} = -\tfrac{7}{18}.$$

Also

$$[(12n+3)a] = [10n+\tfrac{5}{2}] = \tfrac{1}{2},$$
$$[(12n+9)a] = [10n+\tfrac{15}{2}] = -\tfrac{1}{2},$$
$$[(12n+27)a] = [(10n+3)a] = \tfrac{1}{2},$$

and so on. Therefore (as in the preceding section) the exponent of $3 = \frac{1}{8}$.

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We also have

$$[(12n+1)a] = [10n+\tfrac{5}{6}] = \tfrac{1}{6},$$
$$[(12n+5)a] = [10n+\tfrac{25}{6}] = \tfrac{5}{6},$$
$$[(12n+7)a] = [10n+\tfrac{35}{6}] = -\tfrac{5}{6},$$
$$[(12n+11)a] = [10n+\tfrac{55}{6}] = -\tfrac{1}{6},$$

so that:—

if p is a prime of the form $12n+1$, its exponent $= \frac{1}{6}\frac{1}{p-1}$,

" " $12n+5$, " $= \frac{1}{6}\frac{5p+1}{p^2-1}$,

" " $12n+7$, " $= -\frac{1}{6}\frac{5p-1}{p^2-1}$,

" " $12n+11$, " $= -\frac{1}{6}\frac{1}{p+1}$.

The theorem therefore gives

$$2^{-\frac{7}{18}}3^{\frac{1}{6}}5^{\frac{1}{6}\frac{25+1}{5^2-1}}7^{-\frac{1}{6}\frac{35-1}{7^2-1}}11^{-\frac{1}{6}\frac{1}{11+1}}13^{\frac{1}{6}\frac{1}{13-1}}\ldots$$
$$= \frac{\sin\frac{5}{12}\pi\,\Gamma^2(\frac{5}{12})}{(2e^\gamma)^{\frac{1}{2}}\pi^{\frac{3}{2}}};$$

whence, raising each side to the sixth power,

$$5^{\frac{25+1}{5^2-1}}7^{-\frac{35-1}{7^2-1}}11^{-\frac{1}{11+1}}13^{\frac{1}{13-1}}17^{\frac{35+1}{17^2-1}}\ldots$$
$$= \frac{2^{\frac{1}{2}}\sin^6\frac{5}{12}\pi\,\Gamma^{12}(\frac{5}{12})}{3^{\frac{1}{2}}e^{3\gamma}\pi^3} = \frac{2^6K_0^6K_1^{-4}}{3^{\frac{1}{2}}\pi^3e^{3\gamma}}.$$

In the product, the exponent of any prime p is

$$\frac{1}{p-1},\quad \frac{5p+1}{p^2-1},\quad -\frac{5p-1}{p^2-1},\quad -\frac{1}{p+1},$$

according as p is of the form $12n+1$, $12n+5$, $12n+7$, or $12n+11$.

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§ 28. It is evident from these particular cases that, in the general case, where $a = \frac{m}{q}$, a proper fraction in its lowest terms, then, in the formula derived directly from the general theorem of § 19, the primes of the forms $2qn+1$, $2qn+\alpha$, $2qn+\beta$, ..., $2qn+2q-1$, where $1, \alpha, \beta, \ldots, 2q-1$ are the numbers less than $2q$ and prime to it, will all have exponents of different forms.

Formulæ obtained by combining the results for $a = \frac{1}{4}$ *and* $a = \frac{3}{4}$, §§ 29–33.

§ 29. Since

$$\frac{3}{p-1} + \frac{1}{p-1} = \frac{4}{p-1},$$

$$\frac{p+3}{p^2-1} + \frac{3p+1}{p^2-1} = \frac{4}{p-1},$$

$$-\frac{p-3}{p^2-1} - \frac{3p-1}{p^2-1} = -\frac{4}{p+1},$$

$$-\frac{3}{p+1} - \frac{1}{p+1} = -\frac{4}{p+1},$$

we find, by multiplying together the formulæ of §§ 24 and 25, and extracting the fourth root,

$$3^{\frac{1}{3-1}}5^{-\frac{1}{5+1}}7^{-\frac{1}{7+1}}11^{\frac{1}{11-1}}13^{-\frac{1}{13+1}}\ldots = \frac{2^4(\sqrt{2}-1)\,K_{\frac{1}{2}}^2}{\pi^2 e^\gamma},$$

the exponent of a prime p in the product being $\frac{1}{p-1}$ or $-\frac{1}{p+1}$ according as p is of the forms $8n+1$ and $8n+3$, or of the forms $8n+5$ and $8n+7$. This result has been already obtained in § 6.

Similarly by dividing one of the formulæ in §§ 24 and 25 by the other, we obtain M. Rogel's first formula (§§ 2 and 3).

§ 30. Since

$$\frac{p+3}{p^2-1} = -\frac{3}{p+1} + \frac{4p}{p^2-1},$$

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and $$-\frac{p-3}{p^2-1}=\frac{3}{p-1}-\frac{4p}{p^2-1},$$

the product in § 24 may be written

$$3^{-\frac{8}{8+1}}5^{\frac{8}{5-1}}7^{-\frac{8}{7+1}}11^{-\frac{8}{11+1}}13^{\frac{8}{13-1}}\ldots$$

$$\times 3^{\frac{4.8}{8^2-1}}5^{-\frac{4.5}{5^2-1}}11^{\frac{4.11}{11^2-1}}13^{-\frac{4.13}{13^2-1}}\ldots,$$

the second line of which contains only primes of the forms $8n+3$ and $8n+5$.

The first line, by § 21,

$$=\left(\frac{4K_0^2}{\pi^2e^7}\right)^8,$$

and substituting this value, the formula of § 24 becomes

$$\frac{2^6K_0^6}{\pi^6e^{21}}\,3^{\frac{4.8}{8^2-1}}5^{-\frac{4.5}{5^2-1}}11^{\frac{4.11}{11^2-1}}\ldots=\frac{2^{10}(\sqrt{2}-1)^2K_0^2K_2^4}{\pi^6e^{21}},$$

giving

$$3^{\frac{8}{8^2-1}}5^{-\frac{5}{5^2-1}}11^{\frac{11}{11^2-1}}13^{-\frac{13}{13^2-1}}\ldots=\frac{2(\sqrt{2}-1)^4K_2}{K_0},$$

where, in the product, only primes of the forms $8n+3$ and $8n+5$ occur, the sign of the exponent being positive in the former case and negative in the latter.

This result has been already obtained in § 7.

§ 31. In a similar manner we may deduce the formula of § 8 from § 24, for

$$\frac{3}{p-1}=-\frac{1}{p-1}+\frac{4}{p-1},$$

$$\frac{p+3}{p^2-1}=\frac{1}{p+1}+\frac{4}{p^2-1},$$

$$-\frac{p-3}{p^2-1}=-\frac{1}{p-1}+\frac{4}{p^2-1},$$

$$-\frac{3}{p+1}=\frac{1}{p+1}-\frac{4}{p+1},$$

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so that the formula of § 24 may be written

$$3^{\frac{1}{3+1}}5^{-\frac{1}{5-1}}7^{\frac{1}{7+1}}11^{\frac{1}{11+1}}13^{-\frac{1}{13-1}}\ldots$$

$$\times 7^{-\frac{4}{7+1}}17^{\frac{4}{17-1}}\ldots 3^{\frac{4}{3^2-1}}5^{\frac{4}{5^2-1}}11^{\frac{4}{11^2-1}}\ldots = \frac{2^{10}(\sqrt{2}-1)^2K_0^2K_2^4}{\pi^6 e^{3\gamma}}.$$

The first line, by § 21,

$$= \left(\frac{4K_0^2}{\pi^2 e^{\gamma}}\right)^{-1};$$

whence, by substituting this value and extracting the fourth root,

$$7^{-\frac{1}{7+1}}17^{\frac{1}{17-1}}\ldots 3^{\frac{1}{3^2-1}}5^{\frac{1}{5^2-1}}11^{\frac{1}{11^2-1}}\ldots = \frac{2^3(\sqrt{2}-1)^{\frac{1}{2}}K_0K_2}{\pi^{\frac{1}{2}}e^{\frac{\gamma}{4}}},$$

where, in the product, if p is of the form $8n+1$ the exponent is $\frac{1}{p-1}$, if of the form $8n+7$, it is $-\frac{1}{p-1}$, and, if of the forms $8n+3$ and $8n+5$, it is $\frac{1}{p^2-1}$. This is the formula of § 8.

§ 32. It may be remarked that the results obtained in the two preceding sections may be also derived in a similar manner from § 25, viz. the result of § 30 may be derived from § 25 by means of the decompositions

$$\frac{3p+1}{p^2-1} = -\frac{1}{p+1} + \frac{4p}{p^2-1},$$

$$-\frac{3p-1}{p^2-1} = \frac{1}{p-1} - \frac{4p}{p^2-1},$$

and that of § 31 by means of the decompositions

$$\frac{1}{p-1} = -\frac{3}{p-1} + \frac{4}{p-1},$$

$$\frac{3p+1}{p^2-1} = \frac{3}{p+1} + \frac{4}{p^2-1},$$

$$-\frac{3p-1}{p^2-1} = -\frac{3}{p-1} + \frac{4}{p^2-1},$$

$$-\frac{1}{p+1} = \frac{3}{p+1} - \frac{4}{p+1}.$$

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§ 33. The results of §§ 30 and 31 may however be more conveniently and symmetrically obtained by combining the formulæ of §§ 24 and 25. The cube of the formula in § 25 divided by the formula in § 24 gives the result in § 30, and the cube of the formula in § 24 divided by the formula of § 25 gives the result in § 31.

Formulæ obtained by combining the results for $a=\frac{1}{8}$ *and* $a=\frac{3}{8}$, §§ 34–36.

§ 34. Proceeding as in § 29, since

$$\frac{5}{p-1}+\frac{1}{p-1}=\frac{6}{p-1},$$

$$\frac{p+5}{p^2-1}+\frac{5p+1}{p^2-1}=\frac{6}{p-1}, \text{ \&c.},$$

we find, by multiplying together the formulæ in §§ 26 and 27, and extracting the sixth root,

$$5^{\frac{1}{5-1}}7^{-\frac{1}{7+1}}11^{-\frac{1}{11+1}}13^{\frac{1}{13-1}}17^{\frac{1}{17-1}}\ldots=\frac{4.3^{\frac{3}{4}}.K_0'}{\pi^3 e^7},$$

where, in the product, a prime p of the form $12n+1$ or $12n+5$ has the exponent $\frac{1}{p-1}$, and a prime p of the form $12n+7$ or $12n+11$ has the exponent $-\frac{1}{p+1}$. Since the former class of primes are those of the form $4n+1$ and the latter are those of the form $4n+3$, the formula is equivalent to M. Rogel's first formula (§§ 2 and 3).

Similarly by dividing one of the formulæ in §§ 26 and 27 by the other, we obtain the formula of § 5; for the primes of the forms $12n+1$ and $12n+7$ are those of the form $6n+1$, and the primes of the forms $12n+5$ and $12n+11$ are those of the form $6n+5$.

Thus by simple multiplication or division we do not obtain any new formulæ.

§ 35. By raising the formula in § 27 to the fifth power, and dividing by the formula in § 26, we obtain a product in which the exponents are

$$\frac{24p}{p^2-1} \text{ or } -\frac{24p}{p^2-1},$$

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according as p is of the form $12n+5$ or $12n+7$, the value of this product being

$$\frac{2^{12}K_0^{24}}{3^6K_1^{24}},$$

whence, extracting the 24th root,

$$5^{\frac{5}{5^2-1}}7^{-\frac{7}{7^2-1}}17^{\frac{17}{17^2-1}}19^{-\frac{19}{19^2-1}}\ldots=\frac{2^{\frac{1}{2}}K_0}{3^{\frac{1}{4}}K_1}.$$

The product contains only primes of the forms $12n+5$ and $12n+7$, the exponent of any prime p being $\frac{p}{p^2-1}$ or $-\frac{p}{p^2-1}$, according as p is of the former or of the latter form.

§ 36. Similarly, raising the formula in § 26 to the fifth power, and dividing by the formula in § 27, we obtain a product in which the exponents are

$$\frac{24}{p-1},\ \frac{24}{p^2-1},\ \frac{24}{p^3-1},\ -\frac{24}{p+1},$$

according as the prime p is of the form $12n+1$, $12n+5$, $12n+7$, $12n+11$, the value of this product being

$$\frac{2^{36}3^{18}K_0^{24}K_1^{24}}{\pi^{48}e^{24\gamma}}.$$

Extracting the 24th root, we have

$$11^{-\frac{1}{11+1}}13^{\frac{1}{13^2-1}}\ldots5^{\frac{1}{5^2-1}}7^{\frac{1}{7^2-1}}\ldots=\frac{2^{\frac{3}{2}}3^{\frac{1}{4}}K_0K_1}{\pi^2e^{\gamma}},$$

where, in the product, if p is of the form $12n+1$, the exponent is $\frac{1}{p-1}$, if of the form $12n+11$ it is $-\frac{1}{p+1}$, and if of the forms $12n+5$ and $12n+7$, it is $\frac{1}{p^2-1}$.

Combination of formulæ, §§ 37–41.

§ 37. The results obtained in § 30 and § 35 may be written

$$5^{-\frac{5}{5^2-1}}11^{\frac{11}{11^2-1}}13^{-\frac{13}{13^2-1}}19^{\frac{19}{19^2-1}}\ldots=\frac{2(\sqrt{2}-1)^{\frac{1}{2}}K_2}{3^{\frac{1}{4}}K_0},$$

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$$5^{\frac{5}{5^2-1}}7^{-\frac{7}{7^2-1}}17^{\frac{17}{17^2-1}}19^{-\frac{19}{19^2-1}}=\frac{2^{\frac{1}{2}}K_0}{3^{\frac{1}{2}}K_1},$$

where the primes with positive exponents are in the first product those of the form $8n+3$, that is, of the forms

$$24n+11,\ 24n+19;$$

and in the second product those of the form $12n+5$, that is, of the forms

$$24n+5,\ 24n+17;$$

and the primes with negative exponents are, in the first product, those of the form $8n+5$, that is, of the forms

$$24n+5,\ 24n+13;$$

and in the second product, those of the form $12n+7$, that is, of the forms

$$24n+7,\ 24n+19.$$

Thus, multiplying together the two formulæ, we find

$$7^{-\frac{7}{7^2-1}}11^{\frac{11}{11^2-1}}13^{-\frac{13}{13^2-1}}17^{\frac{17}{17^2-1}}\ldots=\frac{2^{\frac{1}{2}}(\sqrt{2}-1)^{\frac{1}{2}}K_0}{3^{\frac{1}{2}}K_1},$$

where the product contains only primes of the forms

$$24n+7,\ 24n+11,\ 24n+13,\ 24n+17,$$

the exponent of a prime p being $\frac{p}{p^2-1}$ if it is of the second or fourth forms, and $-\frac{p}{p^2-1}$ if it is of the first or third forms.

§ 38. Treating in a similar manner the formulæ of § 31 and § 36, we have

$$7^{-\frac{1}{7+1}}17^{\frac{1}{17-1}}\ldots5^{\frac{1}{5^2-1}}11^{\frac{1}{11^2-1}}\ldots=\frac{2^{\frac{1}{2}}(\sqrt{2}-1)^{\frac{1}{2}}K_0K_2}{3^{\frac{1}{2}}\pi^2e^{\gamma}},$$

$$11^{-\frac{1}{11+1}}13^{\frac{1}{13-1}}\ldots5^{\frac{1}{5^2-1}}7^{\frac{1}{7^2-1}}\ldots=\frac{2^{\frac{1}{2}}3^{\frac{1}{2}}K_0K_1}{\pi^2e^{\gamma}}.$$

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The exponent of p is $\frac{1}{p-1}$, in the first product when p is of the forms

$$24n+1,\ 24n+17,$$

and in the second product when p is of the forms

$$24n+1,\ 24n+13.$$

The exponent of p is $-\frac{1}{p+1}$, in the first product when p is of the forms

$$24n+7,\ 24n+23,$$

and in the second product when p is of the forms

$$24n+11,\ 24n+23.$$

The exponent of p is $\frac{1}{p^2-1}$, in the first product when p is of the forms

$$24n+5,\ 24n+11,\ 24n+13,\ 24n+19,$$

and in the second product when p is of the forms

$$24n+5,\ 24n+7,\ 24n+17,\ 24n+19.$$

§ 39. By dividing the first formula by the second, we merely reproduce the result of the preceding section, for the exponents of the primes of the different forms in the quotients are

$$24n+7,\ -\frac{1}{p+1}-\frac{1}{p^2-1}=-\frac{p}{p^2-1},$$

$$24n+11,\ \frac{1}{p+1}+\frac{1}{p^2-1}=\frac{p}{p^2-1},$$

$$24n+13,\ -\frac{1}{p-1}+\frac{1}{p^2-1}=-\frac{p}{p^2-1},$$

$$24n+17,\ \frac{1}{p-1}-\frac{1}{p^2-1}=\frac{p}{p^2-1},$$

and the quotient of the values

$$=\frac{2^{\frac{1}{2}}(\sqrt{2}-1)^{\frac{1}{4}}K_2}{3^{\frac{1}{4}}K_1}.$$

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[illegible]

[illegible]

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is equal to

$$\frac{2^{\frac{1}{2}}3^{\frac{1}{4}}(\sqrt{2}-1)^{\frac{1}{2}}K_1K_2}{\pi^2e^{\gamma}}.$$

§ 42. It may be remarked that if we express the primes in M. Rogel's two formulæ of § 2 by means of the modulus 12, they assume the forms

$$(12n+1,\ 12n+5)^{\frac{1}{p-1}}\ (12n+7,\ 12n+11)^{-\frac{1}{p+1}} = \frac{2^3.3^{\frac{1}{2}}K_0^2}{\pi^2e^{\gamma}},$$

$$(12n+1,\ 12n+7)^{\frac{1}{p-1}}\ (12n+5,\ 12n+11)^{-\frac{1}{p+1}} = \frac{2.3.K_1^2}{\pi^2e^{\gamma}}:$$

the meaning of the notation being that in the first product a prime p of the forms $12n+1$ or $12n+5$ has the exponent $\frac{1}{p+1}$, and so on.

Dividing the first formula by the second, we find

$$(12n+5)^{\frac{1}{p-1}+\frac{1}{p+1}}(12n+7)^{-\frac{1}{p+1}-\frac{1}{p-1}} = \frac{2K_0^2}{3^{\frac{1}{2}}K_1^2},$$

which is equivalent to the result already obtained in § 35. M. Rogel, in the article referred to, divides one of his formulæ by the other, but his result is rendered inexact by a slight slip in the work.

The cases $a = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, §§ 43–51.

§ 43. It is interesting to consider in detail a case in which the denominator of a is a prime number greater than 3. Supposing the denominator therefore to be 5, and putting $a = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, we have the formulæ:

a	$= \frac{1}{5}$,	a	$= \frac{4}{5}$,
$[(10n+1)a]$	$= \frac{4}{5}$,	$[(10n+1)a]$	$= \frac{1}{5}$,
$[(10n+3)a]$	$= \frac{2}{5}$,	$[(10n+3)a]$	$= \frac{3}{5}$,
$[(10n+7)a]$	$= -\frac{3}{5}$,	$[(10n+7)a]$	$= -\frac{2}{5}$,
$[(10n+9)a]$	$= -\frac{4}{5}$;	$[(10n+9)a]$	$= -\frac{1}{5}$;
a	$= \frac{2}{5}$,	a	$= \frac{3}{5}$,
$[(10n+1)a]$	$= \frac{3}{5}$,	$[(10n+1)a]$	$= \frac{2}{5}$,
$[(10n+3)a]$	$= -\frac{1}{5}$,	$[(10n+3)a]$	$= -\frac{4}{5}$,
$[(10n+7)a]$	$= \frac{1}{5}$,	$[(10n+7)a]$	$= \frac{4}{5}$,
$[(10n+9)a]$	$= -\frac{3}{5}$,	$[(10n+9)a]$	$= -\frac{3}{5}$.

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These formulæ are applicable to all uneven primes, but the exponent of 2 has to be calculated separately. For this purpose we find:

$$a=\tfrac{1}{5};\ [2a]=\tfrac{2}{5},\ [4a]=\tfrac{1}{5},\ [8a]=-\tfrac{2}{5},\ [16a]=-\tfrac{1}{5},$$
$$a=\tfrac{2}{5};\ [2a]=\tfrac{1}{5},\ [4a]=-\tfrac{2}{5},\ [8a]=-\tfrac{1}{5},\ [16a]=\tfrac{2}{5},$$
$$a=\tfrac{3}{5};\ [2a]=-\tfrac{1}{5},\ [4a]=\tfrac{2}{5},\ [8a]=\tfrac{1}{5},\ [16a]=-\tfrac{2}{5},$$
$$a=\tfrac{4}{5};\ [2a]=-\tfrac{2}{5},\ [4a]=-\tfrac{1}{5},\ [8a]=\tfrac{2}{5},\ [16a]=\tfrac{1}{5}.$$

The exponent of 5 is zero in all four cases.

§ 44. Taking $a=\frac{1}{5}$, we see that if p is of the form $10n+1$, its exponent

$$=\tfrac{4}{5}\frac{1}{p-1};$$

if p is of the form $10n+3$, it

$$=\tfrac{2}{5}\frac{1}{p}-\tfrac{1}{5}\frac{1}{p^3}-\tfrac{2}{5}\frac{1}{p^5}+\tfrac{1}{5}\frac{1}{p^7}+\tfrac{2}{5}\frac{1}{p^9}-\text{&c.},$$
$$=\tfrac{1}{5}\frac{2p^3-4p^2-2p+4}{p^4-1}=\tfrac{1}{5}\frac{2p-4}{p^2+1};$$

if p is of the form $10n+7$, it

$$=\tfrac{1}{5}\frac{-2p^3-4p^2+2p+4}{p^4-1}=-\tfrac{1}{5}\frac{2p+4}{p^2+1};$$

and, if p is of the form $10n+11$, it

$$=-\tfrac{4}{5}\frac{1}{p+1}.$$

The exponent of 2

$$=\tfrac{1}{2}\cdot\tfrac{2}{5}+\tfrac{1}{4}\cdot\tfrac{1}{5}-\tfrac{1}{8}\cdot\tfrac{2}{5}-\tfrac{1}{16}\tfrac{1}{15}+\tfrac{1}{32}\cdot\tfrac{2}{5}+\text{&c.}$$
$$=\tfrac{1}{5}\frac{3.2+1}{2^2+1}=\tfrac{7}{25}.$$

Thus corresponding to $a=\frac{1}{5}$, we have the formula

$$(10n+1)^{\frac{1}{5}\frac{4}{p-1}}(10n+3)^{\frac{1}{5}\frac{2p-4}{p^2+1}}(10n+7)^{-\frac{1}{5}\frac{2p+4}{p^2+1}}(10n+11)^{-\frac{1}{5}\frac{4}{p+1}}$$
$$=\frac{\sin\frac{1}{10}\pi\,\Gamma^2(\frac{1}{10})}{2^{\frac{2}{5}}(2e^\gamma)^{\frac{4}{5}}\pi^{\frac{2}{5}}}=\frac{\Gamma(\frac{1}{10})}{2^{\frac{2}{5}}\pi^{\frac{4}{5}}e^{\frac{1}{5}\gamma}\Gamma(\frac{9}{10})},$$

the notation being tne same as in § 42, *i.e.*, so that any prime p of the form $10n+1$ has the exponent $\frac{1}{5}\frac{4}{p-1}$, and so on.

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§ 45. Similarly, corresponding to $a=\frac{2}{5}$, we find

$$(10n+1)^{\frac{1}{5}\frac{3}{p-1}}(10n+3)^{-\frac{1}{5}\frac{p+3}{p^2+1}}(10n+7)^{\frac{1}{5}\frac{p-3}{p^2+1}}(10n+11)^{-\frac{1}{5}\frac{3}{p+1}}$$

$$=\frac{2^{\frac{1}{5}}\sin\frac{1}{5}\pi\,\Gamma^2(\frac{1}{5})}{(2e^\gamma)^{\frac{1}{5}}\pi^{\frac{2}{5}}}=\frac{\Gamma(\frac{1}{5})}{2^{\frac{14}{5}}\pi^{\frac{4}{5}}e^{\frac{1}{5}\gamma}\Gamma(\frac{4}{5})},$$

and, corresponding to $a=\frac{3}{5}$ and $a=\frac{4}{5}$,

$$(10n+1)^{\frac{1}{5}\frac{2}{p-1}}(10n+3)^{-\frac{1}{5}\frac{4p+2}{p^2+1}}(10n+7)^{\frac{1}{5}\frac{4p-2}{p^2+1}}(10n+11)^{-\frac{1}{5}\frac{2}{p+1}}$$

$$=\frac{\sin\frac{3}{10}\pi\,\Gamma^2(\frac{3}{10})}{2^{\frac{4}{5}}(2e^\gamma)^{\frac{3}{5}}\pi^{\frac{2}{5}}}=\frac{\Gamma(\frac{3}{10})}{2^{\frac{14}{5}}\pi^{\frac{3}{5}}e^{\frac{3}{5}\gamma}\Gamma(\frac{7}{10})},$$

$$(10n+1)^{\frac{1}{5}\frac{1}{p-1}}(10n+3)^{\frac{1}{5}\frac{3p-1}{p^2+1}}(10n+7)^{-\frac{1}{5}\frac{3p+1}{p^2+1}}(10n+11)^{-\frac{1}{5}\frac{1}{p+1}}$$

$$=\frac{2^{\frac{2}{5}}\sin\frac{2}{5}\pi\,\Gamma^2(\frac{2}{5})}{(2e^\gamma)^{\frac{1}{5}}\pi^{\frac{2}{5}}}=\frac{2^{\frac{2}{5}}\Gamma(\frac{2}{5})}{\pi^{\frac{1}{5}}e^{\frac{1}{5}\gamma}\Gamma(\frac{3}{5})}.$$

§ 46. Multiplying together the first and last of the four formulæ, we find

$$(10n+1)^{\frac{1}{p-1}}(10n+3)^{\frac{p-1}{p^2+1}}(10n+7)^{-\frac{p+1}{p^2+1}}(10n+11)^{-\frac{1}{p+1}}$$

$$=\frac{\Gamma(\frac{1}{10})\Gamma(\frac{2}{5})}{2\pi e^\gamma\Gamma(\frac{9}{10})\Gamma(\frac{3}{5})},$$

and, multiplying together the second and fourth,

$$(10n+1)^{\frac{1}{p-1}}(10n+3)^{-\frac{p+1}{p^2+1}}(10n+7)^{\frac{p-1}{p^2+1}}(10n+11)^{-\frac{1}{p+1}}$$

$$=\frac{\Gamma(\frac{1}{5})\Gamma(\frac{3}{10})}{2\pi e^\gamma\Gamma(\frac{4}{5})\Gamma(\frac{7}{10})}.$$

Dividing the first of these two formulæ by the second, we obtain the result

$$(10n+3)^{\frac{2p}{p^2+1}}(10n+7)^{-\frac{2p}{p^2+1}}=\frac{\Gamma(\frac{1}{10})\Gamma(\frac{2}{5})\Gamma(\frac{1}{5})\Gamma(\frac{7}{10})}{\Gamma(\frac{1}{5})\Gamma(\frac{3}{10})\Gamma(\frac{3}{5})\Gamma(\frac{9}{10})}.$$

§ 47. By putting $n=5$ and $n=\frac{1}{10}$ in the formula

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)$$

$$=\Gamma(nx)(2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nx},$$

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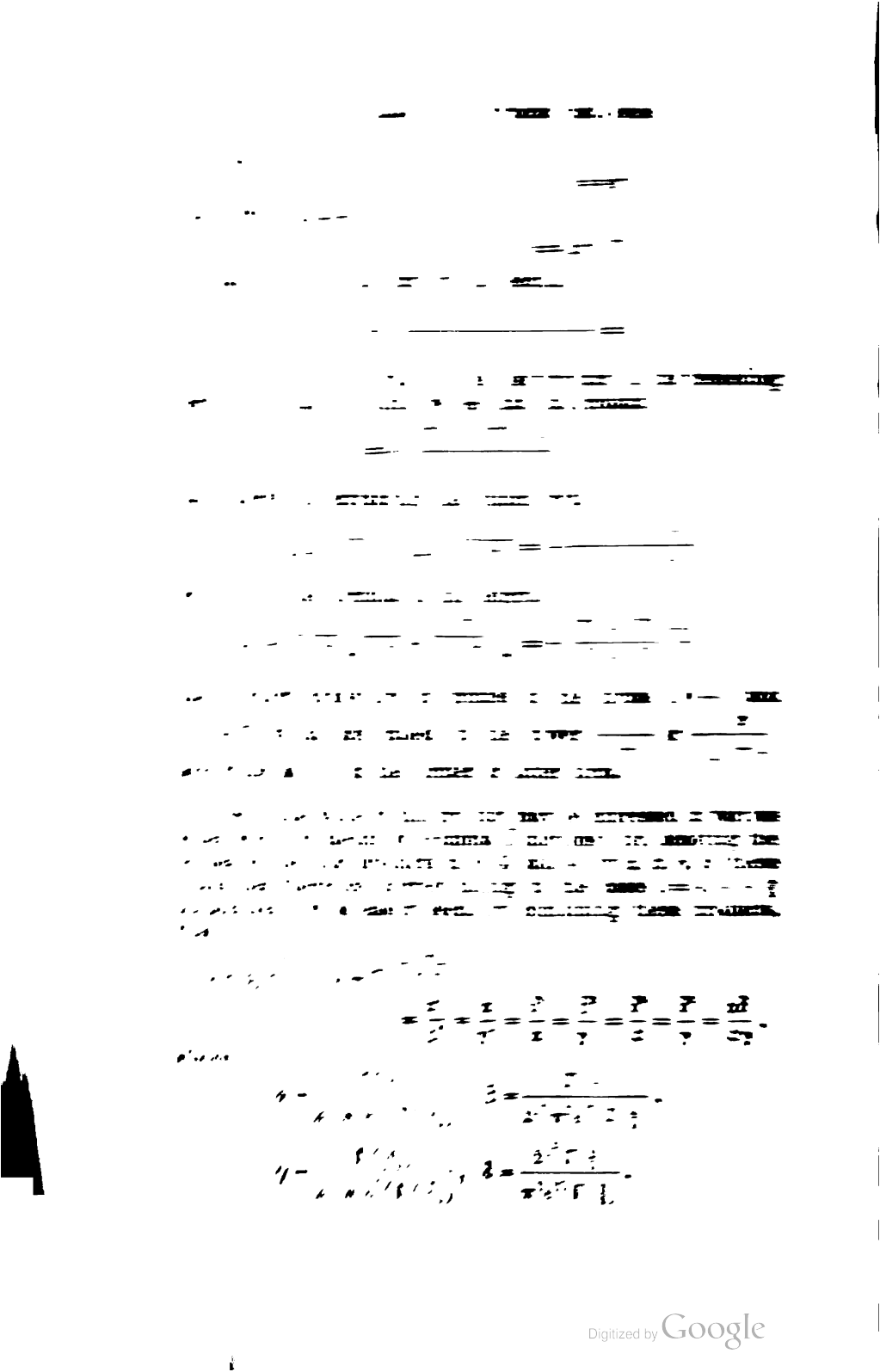

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§ 49. These values of the product imply the relations $\alpha = \beta\delta$, $\beta = \gamma\delta$, giving $\beta^2 = \alpha\gamma$ and $\alpha = \gamma\delta^2$. The relations $\alpha = \beta\delta = \gamma\delta^2$ when written at length are

$$\frac{\Gamma(\frac{1}{10})}{2^{\frac{4}{5}}\Gamma(\frac{9}{10})} = \frac{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{2^{\frac{4}{5}}\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})} = \frac{\Gamma(\frac{3}{10})\Gamma^2(\frac{2}{5})}{\Gamma(\frac{7}{10})\Gamma^2(\frac{3}{5})}.$$

To verify these equalities we notice that by putting $x = \frac{1}{10}$, $\frac{1}{5}$, $\frac{3}{10}$, $\frac{2}{5}$ in the formula

$$\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \pi^{\frac{1}{2}}2^{1-2x}\Gamma(2x),$$

we have

$$\Gamma(\tfrac{1}{10})\Gamma(\tfrac{6}{10}) = \pi^{\frac{1}{2}}2^{\frac{4}{5}}\Gamma(\tfrac{1}{5}),$$
$$\Gamma(\tfrac{1}{5})\Gamma(\tfrac{7}{10}) = \pi^{\frac{1}{2}}2^{\frac{3}{5}}\Gamma(\tfrac{2}{5}),$$
$$\Gamma(\tfrac{3}{10})\Gamma(\tfrac{4}{5}) = \pi^{\frac{1}{2}}2^{\frac{2}{5}}\Gamma(\tfrac{3}{5}),$$
$$\Gamma(\tfrac{2}{5})\Gamma(\tfrac{9}{10}) = \pi^{\frac{1}{2}}2^{\frac{1}{5}}\Gamma(\tfrac{4}{5}).$$

These formulæ enable us to express $\Gamma(\frac{1}{10})$, $\Gamma(\frac{3}{10})$, &c. in terms of $\Gamma(\frac{1}{5})$, $\Gamma(\frac{2}{5})$, &c., viz., we have

$$\Gamma(\tfrac{1}{10}) = 2^{\frac{4}{5}}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})}, \quad \Gamma(\tfrac{3}{10}) = 2^{\frac{2}{5}}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{4}{5})},$$

$$\Gamma(\tfrac{7}{10}) = 2^{\frac{3}{5}}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})}, \quad \Gamma(\tfrac{9}{10}) = 2^{\frac{1}{5}}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{2}{5})},$$

and by substituting these values in the first and third of the equalities, they become equal to the second.

50. It may be remarked that if we express, by means of the above formulæ, the value of the product in § 47 wholly in terms of $\Gamma(\frac{1}{5})$, $\Gamma(\frac{2}{5})$, &c., we find

$$(10n+3)^{\frac{p}{p^2+1}}(10n+7)^{-\frac{p}{p^2+1}} = \frac{2^{\frac{1}{5}}\pi\,\Gamma(\frac{2}{5})}{5^{\frac{1}{2}}\Gamma(\frac{1}{5})\Gamma^2(\frac{3}{5})}.$$

We may also deduce this result from § 48, for

$$\frac{\delta^3}{\beta} = \frac{2^{\frac{4}{5}}\Gamma^3(\frac{2}{5})\Gamma(\frac{4}{5})}{\Gamma^3(\frac{3}{5})\Gamma(\frac{1}{5})},$$

which, when divided by

$$2^{-3}\pi^{-2}5^{\frac{1}{2}}\Gamma(\tfrac{1}{5})\Gamma(\tfrac{2}{5})\Gamma(\tfrac{3}{5})\Gamma(\tfrac{4}{5}) = 1,$$

is

$$= \frac{2^{\frac{17}{5}}\pi^3\Gamma^3(\frac{2}{5})}{5^{\frac{1}{2}}\Gamma^3(\frac{1}{5})\Gamma^4(\frac{3}{5})}.$$

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If we multiply (instead of dividing) by the above expression, we find

$$(10n+3)^{\frac{p}{p^2+1}}(10n+7)^{-\frac{p}{p^2+1}} = \frac{5^{\frac{1}{2}}\Gamma^2(\frac{2}{5})\,\Gamma(\frac{4}{5})}{2^{\frac{3}{2}}\pi\Gamma(\frac{3}{5})}.$$

It will be noticed that the prime numbers which occur in this product are those which (in the decimal system) end in 3 or 7.

§ 51. If we multiply together the first two results in § 46, we find

$$(10n+1)^{\frac{2}{p-1}}(10n+3)^{-\frac{2}{p^2+1}}(10n+7)^{-\frac{2}{p^2+1}}(10n+9)^{-\frac{2}{p+1}}$$

$$= \frac{\Gamma(\frac{1}{10})\,\Gamma(\frac{3}{10})\,\Gamma(\frac{1}{5})\,\Gamma(\frac{2}{5})}{2^3\pi^3 e^{3\gamma}\,\Gamma(\frac{7}{10})\,\Gamma(\frac{9}{10})\,\Gamma(\frac{3}{5})\,\Gamma(\frac{4}{5})},$$

$$= \frac{\Gamma^2(\frac{1}{10})\,\Gamma^2(\frac{3}{10})}{2^2 5^{\frac{1}{2}}\pi^2 e^{2\gamma}\,\Gamma^2(\frac{3}{5})\,\Gamma^2(\frac{4}{5})} = \frac{5^{\frac{1}{2}}\Gamma^2(\frac{1}{5})\,\Gamma^2(\frac{2}{5})}{2^2\pi^2 e^{2\gamma}\,\Gamma^2(\frac{7}{10})\,\Gamma^2(\frac{9}{10})},$$

whence

$$(10n+1)^{\frac{1}{p-1}}(10n+3,\,7)^{-\frac{1}{p^2+1}}(10n+11)^{-\frac{1}{p+1}}$$

$$= \frac{\Gamma(\frac{1}{10})\,\Gamma(\frac{8}{10})}{2.5^{\frac{1}{2}}\pi e^{\gamma}\,\Gamma(\frac{3}{5})\,\Gamma(\frac{4}{5})} = \frac{5^{\frac{1}{2}}\Gamma(\frac{1}{5})\,\Gamma(\frac{2}{5})}{2\pi e^{\gamma}\,\Gamma(\frac{7}{10})\,\Gamma(\frac{9}{10})}$$

$$= \frac{2^{\frac{1}{2}}\Gamma(\frac{1}{5})}{5^{\frac{1}{4}}e^{\gamma}\,\Gamma(\frac{3}{5})\Gamma^2(\frac{4}{5})} = \frac{5^{\frac{1}{4}}\Gamma^2(\frac{1}{5})\,\Gamma(\frac{2}{5})}{2^{\frac{3}{2}}\pi^2 e^{\gamma}\,\Gamma(\frac{4}{5})}.$$

In this product a prime p, if of the form $10n+1$, has the exponent $\frac{1}{p-1}$; if of the form $10n+11$, the exponent $-\frac{1}{p+1}$; and if of either of the forms $10n+3$ or $10n+7$, the exponent $-\frac{1}{p^2+1}$.

The cases $a = \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}$, §§ 52–61.

§ 52. It is interesting also to consider the form of the resulting products given by the general formula in the case of

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a prime having a greater denominator than 5. Taking therefore the case when the denominator is 7, we first form the following scheme corresponding to the six values of a.

a	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$
$[(14n+1)a]$	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
$[(14n+3)a]$	$\frac{4}{7}$	$\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{5}{7}$	$\frac{6}{7}$	$\frac{3}{7}$
$[(14n+5)a]$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	$-\frac{4}{7}$	$\frac{5}{7}$
$[(14n+9)a]$	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{6}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$	$-\frac{5}{7}$
$[(14n+11)a]$	$-\frac{4}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{5}{7}$	$-\frac{6}{7}$	$-\frac{3}{7}$
$[(14n+13)a]$	$-\frac{6}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$

The cycles of residues, with respect to the modulus 14, of the first, second, third, &c. powers of the numbers of the forms $14n+1$, $14n+3$, &c., are shown below:—

$14n+1$; 1,
$14n+3$; 3, 9, 13, 11, 5, 1,
$14n+5$; 5, 11, 13, 9, 3, 1,
$14n+9$; 9, 11, 1,
$14n+11$; 11, 9, 1,
$14n+13$; 13, 1.

§ 53. As an example of the application of these results, consider the exponent of a prime p, of the form $14n+3$, in the case $a=\frac{1}{7}$. This exponent

$$= \frac{1}{p}[pa] + \frac{1}{p^2}[p^2a] + \frac{1}{p^3}[p^3a] + \&c.$$

$$= \tfrac{4}{7}\frac{1}{p} - \tfrac{3}{7}\frac{1}{p^2} - \tfrac{6}{7}\frac{1}{p^3} - \tfrac{4}{7}\frac{1}{p^4} + \tfrac{3}{7}\frac{1}{p^5} + \tfrac{6}{7}\frac{1}{p^6} + \tfrac{4}{7}\frac{1}{p^7} - \&c.$$

$$= \tfrac{1}{7}\frac{4p^3 - 2p - 6}{p^3 + 1}.$$

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Calculating in a similar manner the exponents of the primes of the different forms, we find them to be as follows for the six values of a:—

	$a=\frac{1}{7}$;	$a=\frac{2}{7}$;	$a=\frac{3}{7}$;
$p=14n+1$;	$\frac{1}{7}\frac{6}{p-1}$,	$\frac{1}{7}\frac{5}{p-1}$,	$\frac{1}{7}\frac{4}{p-1}$,
$p=14n+3$;	$\frac{1}{7}\frac{4p^2-2p-6}{p^3+1}$,	$\frac{1}{7}\frac{p^2+3p-5}{p^3+1}$,	$\frac{1}{7}\frac{-2p^2-6p-4}{p^3+1}$,
$p=14n+5$;	$\frac{1}{7}\frac{2p^2-4p-6}{p^3+1}$,	$\frac{1}{7}\frac{-3p^2-p-5}{p^3+1}$,	$\frac{1}{7}\frac{6p^2+2p-4}{p^3+1}$,
$p=14n+9$;	$\frac{1}{7}\frac{-2p^2-4p+6}{p^3-1}$,	$\frac{1}{7}\frac{3p^2-p+5}{p^3-1}$,	$\frac{1}{7}\frac{-6p^2+2p+4}{p^3-1}$,
$p=14n+11$;	$\frac{1}{7}\frac{-4p^2-2p+6}{p^3-1}$,	$\frac{1}{p}\frac{-p^2+3p+5}{p^3-1}$,	$\frac{1}{7}\frac{2p^2-6p+4}{p^3-1}$,
$p=14n+13$;	$\frac{1}{7}\frac{-6}{p+1}$,	$\frac{1}{7}\frac{-5}{p+1}$,	$\frac{1}{7}\frac{-4}{p+1}$;

	$a=\frac{4}{7}$;	$a=\frac{5}{7}$;	$a=\frac{6}{7}$;
$p=14n+1$;	$\frac{1}{7}\frac{3}{p-1}$,	$\frac{1}{7}\frac{2}{p-1}$,	$\frac{1}{7}\frac{1}{p-1}$,
$p=14n+3$;	$\frac{1}{7}\frac{-5p^2-p-3}{p^3+1}$,	$\frac{1}{7}\frac{6p^2+4p-2}{p^3+1}$,	$\frac{1}{7}\frac{3p^2-5p-1}{p^3+1}$,
$p=14n+5$;	$\frac{1}{7}\frac{p^2+5p-3}{p^3+1}$,	$\frac{1}{7}\frac{-4p^2-6p-2}{p^3+1}$,	$\frac{1}{7}\frac{5p^2-3p-1}{p^3+1}$,
$p=14n+9$;	$\frac{1}{7}\frac{-p^2+5p+3}{p^3-1}$,	$\frac{1}{7}\frac{4p^2-6p+2}{p^3-1}$,	$\frac{1}{7}\frac{-5p^2-3p+1}{p^3-1}$,
$p=14n+11$;	$\frac{1}{7}\frac{5p^2-p+3}{p^3-1}$,	$\frac{1}{7}\frac{-6p^2+4p+2}{p^3-1}$,	$\frac{1}{7}\frac{-3p^2-5p+1}{p^3-1}$,
$p=14n+13$;	$\frac{1}{7}\frac{-3}{p+1}$,	$\frac{1}{7}\frac{-2}{p+1}$,	$\frac{1}{7}\frac{-1}{p+1}$.

§ 54. The values of the products are given by the formula

$$\frac{\Gamma(\frac{1}{2}a)}{2^{\frac{1}{2}[2a]+\frac{1}{4}[4a]+\frac{1}{8}[8a]+\cdots}(2\pi e^{\gamma})^{1-a}\Gamma(1-\frac{1}{2}a)}.$$

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With respect to the exponent of 2, since $16 \equiv 2$, mod. 14, we have only to determine the values of $[2a]$, $[4a]$, $[8a]$. These are as follows:—

a	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$
$[2a]$	$\frac{5}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{5}{7}$
$[4a]$	$\frac{3}{7}$	$-\frac{1}{7}$	$-\frac{5}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$
$[8a]$	$-\frac{1}{7}$	$\frac{5}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{5}{7}$	$\frac{1}{7}$

For example, when $a = \frac{1}{7}$, the exponent

$$\frac{1}{2}[2a] + \frac{1}{4}[4a] + \frac{1}{8}[8a] + \&c.$$
$$= \tfrac{5}{7}\cdot\tfrac{1}{2} + \tfrac{3}{7}\cdot\tfrac{1}{4} = \tfrac{1}{7}\cdot\tfrac{1}{8} + \tfrac{5}{7}\cdot\tfrac{1}{16} + \&c.$$
$$= \tfrac{5}{7}\cdot\tfrac{4}{7} + \tfrac{3}{7}\cdot\tfrac{2}{7} - \tfrac{1}{7}\cdot\tfrac{1}{7} = \tfrac{25}{49}.$$

In this way the exponents corresponding to the cases $a = \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}$ are found to be $\frac{25}{49}, \frac{15}{49}, -\frac{8}{49}, \frac{9}{49}, -\frac{15}{49}, -\frac{25}{49}$ respectively.

The values of the products in the six cases are therefore as follows:—

$$a = \tfrac{1}{7};\ \frac{\Gamma(\frac{1}{14})}{2^{\frac{12}{7}}\pi^{\frac{6}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{13}{14})} = \frac{(\sin\frac{1}{14}\pi)\,\Gamma^2(\frac{1}{14})}{2^{\frac{12}{7}}\pi^{\frac{1}{7}}e^{\frac{6}{7}\gamma}},$$

$$a = \tfrac{2}{7};\ \frac{\Gamma(\frac{1}{7})}{2^{\frac{12}{7}}\pi^{\frac{6}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{6}{7})} = \frac{(\sin\frac{1}{7}\pi)\,\Gamma^2(\frac{1}{7})}{2^{\frac{12}{7}}\pi^{\frac{1}{7}}e^{\frac{6}{7}\gamma}},$$

$$a = \tfrac{3}{7};\ \frac{\Gamma(\frac{3}{14})}{2^{\frac{12}{7}}\pi^{\frac{6}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{11}{14})} = \frac{(\sin\frac{3}{14}\pi)\,\Gamma^2(\frac{3}{14})}{2^{\frac{12}{7}}\pi^{\frac{1}{7}}e^{\frac{6}{7}\gamma}},$$

$$a = \tfrac{4}{7};\ \frac{\Gamma(\frac{2}{7})}{2^{\frac{12}{7}}\pi^{\frac{5}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{5}{7})} = \frac{(\sin\frac{2}{7}\pi)\,\Gamma^2(\frac{2}{7})}{2^{\frac{12}{7}}\pi^{\frac{4}{7}}e^{\frac{6}{7}\gamma}},$$

$$a = \tfrac{5}{7};\ \frac{2^{\frac{1}{7}}\Gamma(\frac{5}{14})}{\pi^{\frac{5}{7}}e^{\frac{5}{7}\gamma}\Gamma(\frac{9}{14})} = \frac{2^{\frac{1}{7}}(\sin\frac{5}{14}\pi)\,\Gamma^2(\frac{5}{14})}{\pi^{\frac{5}{7}}e^{\frac{5}{7}\gamma}},$$

$$a = \tfrac{6}{7};\ \frac{2^{\frac{13}{7}}\Gamma(\frac{3}{7})}{\pi^{\frac{1}{7}}e^{\frac{1}{7}\gamma}\Gamma(\frac{4}{7})} = \frac{2^{\frac{13}{7}}(\sin\frac{3}{7}\pi)\,\Gamma^2(\frac{3}{7})}{\pi^{\frac{5}{7}}e^{\frac{1}{7}\gamma}}.$$

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Thus, for example, the first formula (corresponding to $a=\frac{1}{7}$) written at full length is:—

$$(14n+1)^{\frac{1}{7}\frac{6}{p-1}}(14n+3)^{\frac{1}{7}\frac{4p^2-2p-6}{p^2+1}}(14n+5)^{\frac{1}{7}\frac{2p^2-4p-6}{p^2+1}}$$

$$\times(14n+9)^{\frac{1}{7}\frac{-2p^2-4p+6}{p^2-1}}(14n+11)^{\frac{1}{7}\frac{-4p^2-2p+6}{p^2-1}}(14n+13)^{\frac{1}{7}\frac{-6}{p+1}}$$

$$=\frac{\Gamma(\frac{1}{14})}{2^{\frac{2}{7}}\pi^{\frac{4}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{13}{14})}=\frac{(\sin\frac{1}{14}\pi)\,\Gamma^2(\frac{1}{14})}{2^{\frac{2}{7}}\pi^{\frac{1}{7}}e^{\frac{6}{7}\gamma}\Gamma(\frac{13}{14})},$$

the notation being the same as in § 42.

§ 55. By multiplying together the formulæ corresponding to $a=\frac{1}{7}$ and $a=\frac{6}{7}$, we find that the product in which the exponent of a prime p is

$$\frac{1}{p-1},\ \frac{p^2-p-1}{p^3+1},\ \frac{p^2-p-1}{p^3+1},\ \frac{-p^2-p+1}{p^3-1},\ \frac{-p^2-p+1}{p^3-1},\ -\frac{1}{p+1},$$

according as p is of the forms $14n+1$, 3, 5, 9, 11, 13, is

$$\frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{3}{7})}{2\pi e^{\gamma}\,\Gamma(\frac{13}{14})\,\Gamma(\frac{4}{7})}=\frac{(\sin\frac{1}{7}\pi)\,\Gamma^2(\frac{1}{14})\,\Gamma^2(\frac{3}{7})}{4\pi^3e^{\gamma}}\ \ldots(A).$$

Similarly, by multiplying together the formulæ corresponding to $a=\frac{2}{7}$ and $a=\frac{5}{7}$, and those corresponding to $a=\frac{3}{7}$ and $a=\frac{4}{7}$, we find that the products in which the exponents are

$$\frac{1}{p-1},\ \frac{p^2+p-1}{p^3+1},\ \frac{-p^2-p-1}{p^3+1},\ \frac{p^2-p+1}{p^3-1},\ \frac{-p^2+p+1}{p^3-1},\ -\frac{1}{p+1},$$

$$\frac{1}{p-1},\ \frac{-p^2-p-1}{p^3+1},\ \frac{p^2+p-1}{p^3+1},\ \frac{-p^2+p+1}{p^3-1},\ \frac{p^2-p+1}{p^3-1},\ -\frac{1}{p+1},$$

are
$$\frac{\Gamma(\frac{1}{7})\,\Gamma(\frac{5}{14})}{2\pi e^{\gamma}\,\Gamma(\frac{6}{7})\,\Gamma(\frac{9}{14})}=\frac{(\sin\frac{2}{7}\pi)\,\Gamma^2(\frac{1}{7})\,\Gamma^2(\frac{5}{14})}{4\pi^3e^{\gamma}}\ \ldots(B),$$

and
$$\frac{\Gamma(\frac{3}{14})\,\Gamma(\frac{2}{7})}{2\pi e^{\gamma}\,\Gamma(\frac{11}{14})\,\Gamma(\frac{5}{7})}=\frac{(\sin\frac{3}{7}\pi)\,\Gamma^2(\frac{3}{14})\,\Gamma^2(\frac{2}{7})}{4\pi^3e^{\gamma}}\ \ldots(C),$$

respectively.

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§ 56. By dividing the formula (A) by the formula (B) and extracting the square root, we find that

$$(14n+3)^{-\frac{p}{p^2+1}}(14n+5)^{\frac{p^2}{p^2+1}}(14n+9)^{-\frac{p^2}{p^2-1}}(14n+11)^{-\frac{p}{p^2-1}}$$
$$= \left(\frac{\sin\frac{1}{7}\pi}{\sin\frac{2}{7}\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{3}{7})}{\Gamma(\frac{1}{7})\,\Gamma(\frac{5}{14})} \ldots\ldots\ldots\ldots \left(\frac{A}{B}\right),$$

and, by treating the formulæ (A) and (C) in a similar manner,

$$(14n+3)^{\frac{p^2}{p^2+1}}(14n+5)^{-\frac{p}{p^2+1}}(14n+9)^{-\frac{p}{p^2-1}}(14n+1)^{-\frac{p^2}{p^2-1}}$$
$$= \left(\frac{\sin\frac{1}{7}\pi}{\sin\frac{2}{7}\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{3}{7})}{\Gamma(\frac{3}{14})\,\Gamma(\frac{2}{7})} \ldots\ldots\ldots \left(\frac{A}{C}\right).$$

In these products primes of the forms $14n+1$ and $14n+11$ do not occur.

§ 57. By multiplying (B) and (C) and extracting the square root, we find

$$(14n+1)^{\frac{1}{p-1}}(14n+3,5)^{-\frac{1}{p^2+1}}(14n+9,11)^{\frac{1}{p^2-1}}(14n+13)^{-\frac{1}{p+1}}$$
$$= \frac{(\sin\frac{2}{7}\pi\,\sin\frac{3}{7}\pi)^{\frac{1}{2}}\,\Gamma(\frac{1}{7})\,\Gamma(\frac{3}{14})\,\Gamma(\frac{2}{7})\,\Gamma(\frac{5}{14})}{4\pi^3 e^{\gamma}} \ldots (BC).$$

As in § 51, the meaning of the notation is that any prime p of the form $14n+1$ is to be raised to the power $\frac{1}{p-1}$, that any prime p of the forms $14n+3$ or $14n+5$ is to be raised to the power $-\frac{1}{p^2+1}$, any prime p of the forms $14n+9$ or $14n+11$ to the power $\frac{1}{p^2-1}$, and any prime p of the form $14n+13$ to the power $-\frac{1}{p+1}$.

§ 58. By means of the last formula we may simplify the six product-formulæ obtained in §§ 53 and 54; for, denoting the values of the six products given in § 54, by $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ respectively; and denoting the value of the product in (BC) by Δ, so that $\Delta = (\beta\gamma\delta\varepsilon)^{\frac{1}{2}}$, we may replace the results of §§ 53 and 54 by the scheme:—

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	$p=$	$14n+3$,	$14n+5$,	$14n+9$,	$14n+11$, value,
exp. =		$\frac{4p^2-2p}{p^3+1}$,	$\frac{2p^2-4p}{p^3+1}$,	$\frac{-2p^2-4p}{p^3-1}$,	$\frac{-4p^2-2p}{p^3-1}$, $\frac{\alpha^7}{\Delta^6}$,
"		$\frac{p^2+3p}{p^3+1}$,	$\frac{-3p^2-p}{p^3+1}$,	$\frac{3p^2-p}{p^3-1}$,	$\frac{-p^2+3p}{p^3-1}$, $\frac{\beta^7}{\Delta^5}$,
"		$\frac{-2p^2-6p}{p^3+1}$,	$\frac{6p^2+2p}{p^3+1}$,	$\frac{-6p^2+2p}{p^3-1}$,	$\frac{2p^2-6p}{p^3-1}$, $\frac{\gamma^7}{\Delta^4}$,
"		$\frac{-5p^2-p}{p^3+1}$,	$\frac{p^2+5p}{p^3+1}$,	$\frac{-p^2+5p}{p^3-1}$,	$\frac{5p^2-p}{p^3-1}$, $\frac{\delta^7}{\Delta^3}$,
"		$\frac{6p^2+4p}{p^3+1}$,	$\frac{-4p^2-6p}{p^3+1}$,	$\frac{4p^2-6p}{p^3-1}$,	$\frac{-6p^2+4p}{p^3-1}$, $\frac{\varepsilon^7}{\Delta^2}$,
"		$\frac{3p^2-5p}{p^3+1}$,	$\frac{5p^2-3p}{p^3+1}$,	$\frac{-5p^2-3p}{p^3-1}$,	$\frac{-3p^2-5p}{p^3-1}$, $\frac{\zeta^7}{\Delta}$.

The meaning of the notation is almost obvious, viz., for example, the first line indicates that the product

$$(14n+3)^{\frac{4p^2-2p}{p^3+1}}(14n+5)^{\frac{2p^2-4p}{p^3+1}}(14n+9)^{\frac{-2p^2-4p}{p^3-1}}(14n+11)^{\frac{-4p^2-2p}{p^3-1}}$$

has the value $\frac{\alpha^7}{\Delta^6}$.

§ 59. It is easy to verify that these six products are all derivable from the two $\left(\frac{A}{B}\right)$ and $\left(\frac{A}{C}\right)$ of § 56. We thus see that the six results in §§ 53 and 54, which were obtained directly from the general theorem of § 19, are not independent. In fact, they suffice to assign the values of only three independent products, which in this simplest form may be written

	$p=14n+1$,	$14n+3$, 5,	$14n+9$, 11,	$14n+13$,
exp. =	$\frac{1}{p-1}$,	$-\frac{1}{p^3+1}$,	$\frac{1}{p^3-1}$,	$-\frac{1}{p+1}$;

and	$p=14n+3$,	$14n+5$,	$14n+9$,	$14n+11$,
exp. =	$-\frac{p}{p^3+1}$,	$\frac{p^2}{p^3+1}$,	$-\frac{p^2}{p^3-1}$,	$-\frac{p}{p^3-1}$;
"	$\frac{p^2}{p^3+1}$,	$-\frac{p}{p^3+1}$,	$-\frac{p}{p^3-1}$,	$-\frac{p^2}{p^3-1}$.

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The values of these products, to which the different methods lead, must be equivalent to each other by virtue of the properties of the Gamma Function.

§ 60. Another formula, even simpler than (*BC*) may be derived from (*A*), (*B*), (*C*). For, multiplying together $\left(\frac{A}{B}\right)$ and $\left(\frac{A}{C}\right)$ of § 56, we find

$$(14n+3,\,5)^{\frac{p^2-p}{p^2+1}}(14n+9,\,11)^{\frac{-p^2-p}{p^2-1}}$$

$$= \frac{\sin\frac{1}{7}\pi\,\Gamma^2\left(\frac{1}{14}\right)\Gamma^2\left(\frac{3}{7}\right)}{(\sin\frac{3}{7}\pi\sin\frac{2}{7}\pi)^{\frac{1}{2}}\,\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{5}{14}\right)},$$

and, dividing (*BC*) of § 57 by this formula, we obtain the result

$$(14n+1,\,9,\,11)^{\frac{1}{p-1}}\,(14n+3,\,5,\,13)^{-\frac{1}{p+1}}$$

$$= \frac{\sin\frac{2}{7}\pi\sin\frac{3}{7}\pi\,\Gamma^2\left(\frac{1}{7}\right)\Gamma^2\left(\frac{3}{14}\right)\Gamma^2\left(\frac{3}{7}\right)\Gamma^2\left(\frac{5}{14}\right)}{4\pi^3 e^{\gamma}\sin\frac{1}{7}\pi\,\Gamma^2\left(\frac{1}{14}\right)\Gamma^2\left(\frac{2}{7}\right)}.$$

§ 61. Other results of less interest may of course be obtained by combining in other ways the formulæ (*A*), (*B*), (*C*). Thus we find, by division from $\left(\frac{A}{B}\right)$ and $\left(\frac{A}{C}\right)$,

$$(14n+3)^{\frac{p}{p^2-p+1}}(14n+5)^{-\frac{p}{p^2-p+1}}(14n+9)^{\frac{p}{p^2+p+1}}(14n+11)^{-\frac{p}{p^2+p+1}}$$

$$= \frac{(\sin\frac{2}{7}\pi)^{\frac{1}{2}}\,\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{5}{14}\right)}{(\sin\frac{3}{7}\pi)^{\frac{1}{2}}\,\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{2}{7}\right)}.$$

Discussion of the fundamental formulæ obtained, §§ 62–64.

§ 62. It is interesting to classify, as in § 59, all the independent or fundamental products which have been obtained in this paper by assigning special values to a in the general theorem of § 19.

Putting $a=\frac{m}{q}$, a proper fraction in its lowest terms, and

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first considering the prime values 2, 3, 5, 7 of q, the principal products obtained are:

$q=2$;

$$\begin{array}{c|cc} p= & 4n+1 & 4n+3 \\ \hline \text{exp.} = & \dfrac{1}{p-1} & -\dfrac{1}{p+1} \end{array};$$

$q=3$;

$$\begin{array}{c|cc} p= & 6n+1 & 6n+5 \\ \hline \text{exp.} = & \dfrac{1}{p-1} & -\dfrac{1}{p+1} \end{array};$$

$q=5$;

$$\begin{array}{c|ccc} p= & 10n+1, & 10n+3, 7, & 10n+9, \\ \hline \text{exp.} = & \dfrac{1}{p-1}, & -\dfrac{1}{p^2+1} & , -\dfrac{1}{p+1} \end{array};$$

$$\begin{array}{c|cc} p= & 10n+3, & 10n+7, \\ \hline \text{exp.} = & \dfrac{p}{p^4+1}, & -\dfrac{p}{p^4+1} \end{array};$$

$q=7$;

$$\begin{array}{c|cc} p= & 14n+1, 9, 11, & 14n+3, 5, 13, \\ \hline \text{exp.} = & \dfrac{1}{p-1}, & -\dfrac{1}{p+1} \end{array};$$

$$\begin{array}{c|cccc} p= & 14n+1, & 14n+3, 5, & 14n+9, 11, & 14n+13, \\ \hline \text{exp.} = & \dfrac{1}{p-1}, & -\dfrac{1}{p^3+1}, & \dfrac{1}{p^3-1}, & -\dfrac{1}{p+1} \end{array};$$

$$\begin{array}{c|cccc} p= & 14n+3, & 14n+5, & 14n+9, & 14n+11, \\ \hline \text{exp.} = & -\dfrac{p}{p^3+1}, & \dfrac{p^2}{p^3+1}, & -\dfrac{p^2}{p^3-1}, & -\dfrac{p}{p^3-1} \end{array},$$

and

$$\text{exp.} = \frac{p^2}{p^3+1}, \quad -\frac{p}{p^3+1}, \quad -\frac{p}{p^3-1}, \quad -\frac{p^2}{p^3-1}.$$

Of these last four products, only three are independent.

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§ 63. Considering now the composite values 4 and 6 of q, we have the products:

$q = 4$;

$$p = 8n+1, 3,\ 8n+5, 7,$$

$$\text{exp.} = \frac{1}{p-1}, -\frac{1}{p+1};$$

$$p = 8n+1,\ 8n+3, 5,\ 8n+7,$$

$$\text{exp.} = \frac{1}{p-1}, \frac{1}{p^2-1}, -\frac{1}{p+1};$$

$$p = 8n+3,\ 8n+5,$$

$$\text{exp.} = \frac{p}{p^2-1}, -\frac{p}{p^2-1};$$

$q = 6$;

$$p = 12n+1,\ 12n+5, 7,\ 12n+11,$$

$$\text{exp.} = \frac{1}{p-1}, \frac{1}{p^2-1}, -\frac{1}{p+1};$$

$$p = 12n+5,\ 12n+7,$$

$$\text{exp.} = \frac{p}{p^2-1}, -\frac{p}{p^2-1}.$$

§ 64. Only one of the three products corresponding to $q = 4$ is fundamental or independent, *i.e.*, from any one of these three products, combined with the product for $q = 2$, the other two may be deduced. In the case of $q = 6$, neither of the products is fundamental, *i.e.*, both may be deduced from the products for $q = 2$ and $q = 3$.

Thus, taking the first of the three products for $q = 4$, viz.

$$p = 8n+1,\ 8n+3,\ 8n+5,\ 8n+7,$$

$$\text{exp.} = \frac{1}{p-1}, \frac{1}{p-1}, -\frac{1}{p+1}, -\frac{1}{p+1};$$

and combining it with the product for $q = 2$, viz.

$$\text{exp.} = \frac{1}{p-1}, -\frac{1}{p+1}, \frac{1}{p-1}, -\frac{1}{p+1};$$

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we obtain the product

$$\frac{2p}{p^2-1}, \quad -\frac{2p}{p^4-1},$$

by division, and the product

$$\frac{2}{p-1}, \quad \frac{2}{p^2-1}, \quad -\frac{2}{p^3-1}, \quad -\frac{2}{p+1},$$

by multiplication.

Similarly from the products corresponding to $q=2$ and $q=3$, which may be written

$$p = 12n+1, \quad 12n+5, \quad 12n+7, \quad 12n+11,$$

$$\text{exp.} = \frac{1}{p-1}, \quad \frac{1}{p-1}, \quad -\frac{1}{p+1}, \quad -\frac{1}{p+1};$$

$$\text{exp.} = \frac{1}{p-1}, \quad -\frac{1}{p+1}, \quad \frac{1}{p-1}, \quad -\frac{1}{p+1};$$

we deduce by division and multiplication the two products corresponding to $q=6$.

Values of the thirteen series, §§ 65, 66.

§ 65. The values of the thirteen products in §§ 62 and 63 are given below in terms of elliptic integrals (except in the cases $q=5$ and 7), and of Gamma functions.

The results are expressed as series-formulæ, obtained by taking the logarithms of the product-formulæ. The series and products are convertible into each other at sight, but in giving a system of results for reference it seems preferable to select the series.

(i) If $A_p = \frac{1}{p-1}$, or $-\frac{1}{p+1}$,

according as $p \equiv 1$, or 3, mod. 4,

then $$\Sigma_2^\infty A_p \log p = \log\left\{\frac{4K_0^2}{\pi^2 e^\gamma}\right\} = \log\left\{\frac{\Gamma^4(\frac{1}{4})}{4\pi^2 e^\gamma}\right\}.$$

(ii) If $A_p = \frac{1}{p-1}$, or $-\frac{1}{p+1}$,

according as $p \equiv 1$, or 5, mod. 6,

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then

$$\Sigma_1^\infty A_p \log p = \log\left\{\frac{2.3.K_1^3}{\pi^2 e^\gamma}\right\} = \log\left\{\frac{\Gamma^3(\frac{1}{6})}{2^{\frac{5}{3}}\pi^{\frac{1}{2}}e^\gamma}\right\} = \log\left\{\frac{3^{\frac{3}{2}}\Gamma^3(\frac{1}{3})}{2^{\frac{1}{3}}\pi^4 e^\gamma}\right\}.$$

(iii) If $$A_p = \frac{1}{p-1}, \text{ or } -\frac{1}{p+1},$$

according as $$p \equiv 1, 3, \text{ or } 5, 7, \text{ mod. } 8,$$

then

$$\Sigma_1^\infty A_p \log p = \log\left\{\frac{2^4(\sqrt{2}-1)K_2^2}{\pi^2 e^\gamma}\right\} = \log\left\{\frac{(\sin\frac{1}{8}\pi)^2\Gamma^4(\frac{1}{8})}{2\pi^2 e^\gamma \Gamma^2(\frac{1}{4})}\right\}.$$

(iv) If $$A_p = \frac{1}{p-1}, \frac{1}{p^2-1}, \text{ or } -\frac{1}{p+1},$$

according as $$p \equiv 1, \text{ or } 3, 5, \text{ or } 7, \text{ mod. } 8,$$

then

$$\Sigma_1^\infty A_p \log p = \log\left\{\frac{2^3(\sqrt{2}-1)^{\frac{1}{2}}K_0 K_2}{\pi^2 e^\gamma}\right\} = \log\left\{\frac{\sin\frac{1}{8}\pi\,\Gamma(\frac{1}{4})\Gamma^2(\frac{1}{8})}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}e^\gamma}\right\}.$$

(v) If $$A_p = \frac{p}{p^2-1}, \text{ or } -\frac{p}{p^2-1},$$

according as $$p \equiv 3, \text{ or } 5, \text{ mod. } 8,$$

then

$$\Sigma_1^\infty A_p \log p = \log\left\{\frac{2(\sqrt{2}-1)^{\frac{1}{2}}K_2}{K_0}\right\} = \log\left\{\frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{\Gamma^2(\frac{1}{4})}\right\}.$$

(vi) If $$A_p = \frac{1}{p-1}, -\frac{1}{p^2+1}, \text{ or } -\frac{1}{p+1},$$

according as $$p \equiv 1, \text{ or } 3, 7, \text{ or } 9, \text{ mod. } 10,$$

then

$$\Sigma_1^\infty A_p \log p = \log\left\{\frac{(\sin\frac{1}{5}\pi\sin\frac{2}{5}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{10})\Gamma(\frac{2}{10})\Gamma(\frac{3}{10})\Gamma(\frac{4}{10})}{4\pi^3 e^\gamma}\right\}.$$

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(xii) If $A_p = -\frac{p}{p^3+1}, \frac{p^3}{p^3+1}, -\frac{p^3}{p^3-1}, -\frac{p}{p^3-1},$

according as $p \equiv 3$, or 5, or 9, or 11, mod. 14,

then $$\Sigma_3^\infty A_p \log p = \log\left\{\left(\frac{\sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{6}{14})}{\Gamma(\frac{2}{14})\,\Gamma(\frac{5}{14})}\right\}.$$

(xiii) If

$$A_p = \frac{p^3}{p^3+1}, -\frac{p}{p^3+1}, -\frac{p}{p^3-1}, \text{ or } -\frac{p^3}{p^3-1},$$

according as $p \equiv 3$, or 5, or 9, or 11, mod. 14,

then $$\Sigma_3^\infty A_p \log p = \log\left\{\left(\frac{\sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{3}{14})}{\Gamma(\frac{3}{14})\,\Gamma(\frac{4}{14})}\right\}.$$

§ 66. In all the above series p is to have as values all the primes of the forms specified. The primes which are not of the forms specified do not occur in the series (or product). For example, the formula (v) is

$$\frac{3\log 3}{3^3-1} - \frac{5\log 5}{5^3-1} + \frac{11\log 11}{11^3-1} - \frac{13\log 13}{13^3-1} + \frac{19\log 19}{19^3-1} - \&c.$$

$$= \log\left\{\frac{2\,(\sqrt{2}-1)^{\frac{1}{2}}\,K_2}{K_0}\right\} = \log\left\{\frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{\Gamma^2(\frac{1}{4})}\right\},$$

the primes of the forms $8n+1$ and $8n+7$ not occurring.

The systems of formulæ derived from the general theorem are rendered all the more interesting by the fact that the number of series, involving prime numbers only, which admit of summation is not very great.

General relations between the exponents, §§ 67, 68.

§ 67. It will be observed that in the system of exponents given in § 53, those which, for the same value of a, belong to the primes of the forms $14n+9$, $14n+11$, $14n+13$ differ from those which belong to the primes of the forms $14n+5$, $14n+3$, $14n+1$, respectively, only by a change in the sign of p. A similar relation also holds good in the corresponding

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systems derived from the other values of q. It is easy to prove that this law is true generally for all the systems derived from the general theorem.

Taking the case of $q=7$, it is evident that the relation in question is a consequence of the fact that (i) the results in the lines equidistant from the top and bottom in the scheme in § 52 differ only in sign, and (ii) the residues in the cycle for 5 and 9, for 3 and 11, and for 1 and 13, mod. 14 are alternately complementary and equal.

Now the corresponding results to these hold good generally. For, putting $a=\frac{m}{q}$, a proper fraction in its lowest terms, and supposing $r<2q$ and prime to it,

$$[(2qn+r)\,a]=\left[(2qn+r)\frac{m}{q}\right]=\left[2n+\frac{rm}{q}\right]=\left[\frac{rm}{q}\right],$$

and $$[2qn+(2q-r)\,a]=\left[2n+2-\frac{rm}{q}\right]=\left[2-\frac{rm}{q}\right],$$

the value of which is easily seen to be equal to that of $-\left[\frac{rm}{q}\right]$.

Also, if the cycle of residues of $r, r^2, r^3, \ldots$ with respect to the modulus $2q$ are $r, \alpha, \beta, \ldots$, then evidently the cycle of residues of $2q-r, (2q-r)^2, (2q-r)^3, \ldots$ will be $2q-r, \alpha, 2q-\beta, \ldots$.

Thus the above law holds good with respect to all formulæ derived directly from the general theorem, and therefore also with respect to all the other formulæ which are obtained by their combination. It is observable in all the formulæ of § 65.

§ 68. If we add together the exponents corresponding to any given value of a and to its complementary value $1-a$ in § 53 (as is done in obtaining the formulæ A, B, C of § 55), we notice that all the coefficients of the powers of p in the numerators are equal to ± 1. A proof of this law, which is general, is contained in § 74.

Second form of the general theorem, §§ 69–73.

§ 69. The general theorem of § 19 may also be expressed in a somewhat different form as follows.

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Putting $1-a$ for a in M. Rogel's general theorem (§§ 1 and 18), it becomes

$$\frac{3^{\frac{\sin 3a\pi}{3}}\,5^{\frac{\sin 5a\pi}{5}}\,7^{\frac{\sin 7a\pi}{7}}\ldots}{2^{\frac{\sin 2a\pi}{2}}\,4^{\frac{\sin 4a\pi}{4}}\,6^{\frac{\sin 6a\pi}{6}}\ldots} = \left\{\frac{\cos\frac{1}{2}a\pi\;\Gamma^2\left(\frac{1-a}{2}\right)}{2^a\pi^{1+a}e^{a\gamma}}\right\}^{\frac{1}{\pi}}$$

$$= \left\{\frac{1}{(2\pi e^{\gamma})^a}\,\frac{\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)}\right\}^{\frac{1}{\pi}},$$

in which a must lie between 0 and 2.*

In the quotient on the left-hand side, the exponent of 3, when the numbers are decomposed into their prime factors, is

$$\frac{\sin 3a\pi}{3} - \frac{\sin 6a\pi}{6} + 2\frac{\sin 9a\pi}{9} - \frac{\sin 12a\pi}{12} + \frac{\sin 15a\pi}{15} - \&c.,$$

the coefficient of 5 is

$$\sin\frac{5a\pi}{5} - \frac{\sin 10a\pi}{10} + \frac{\sin 15a\pi}{15} - \frac{\sin 20a\pi}{20} + 2\frac{\sin 25a\pi}{25} - \&c.,$$

and so on, the negative sign corresponding to the even multiples; but the exponent of 2 in the denominator is

$$\frac{\sin 2a\pi}{2} + 2\frac{\sin 4a\pi}{4} + \frac{\sin 6a\pi}{6} + 3\frac{\sin 8a\pi}{8} + \&c.,$$

all the signs being positive.

§ 70. Now by proceeding as in § 16, it can be shown that, if

$$\psi(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \&c.,$$

then

$$\psi(x) = \frac{x}{2}, \quad 0\,(x)\,\pi,$$

$$= \frac{x-2\pi}{2}, \quad \pi\,(x)\,3\pi,$$

$$= \frac{x-4\pi}{4}, \quad 3\pi\,(x)\,4\pi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

* This form of the theorem was noticed in vol. xxiv. of the *Messenger*, p. 3.

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and, in general,

$$\psi(x) = \frac{x - 2n\pi}{2}, \quad (2n-1)(x)(2n+1)\pi,$$

where, as in § 15, $a(x)b$ means that x is intermediate in value to a and b.

Thus, for all positive values of x,

$$\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 4\pi x}{4} - \frac{\sin 6\pi x}{6} + \&c. = \tfrac{1}{2}\pi \{x\},$$

where $\{x\}$ denotes $x - m$, m being the nearest even integer to x.

If x is an integer, $\{x\}$ is supposed to be zero.

§ 71. Substituting for the sine-series in the exponents their numerical values, and extracting the $\left(\frac{\pi}{2}\right)^{\text{th}}$ root of both sides, the formula of § 69 becomes

$$\frac{3^{\frac{1}{3}\{3a\}+\frac{1}{9}\{9a\}+\frac{1}{27}\{27a\}+\ldots}\, 5^{\frac{1}{5}\{5a\}+\frac{1}{25}\{25a\}+\ldots}\ldots}{2^{\frac{1}{2}[2a]+\frac{1}{4}[4a]+\frac{1}{8}[8a]+\ldots}} = \frac{\cos\frac{1}{2}a\pi\,\Gamma^2\left(\frac{1-a}{2}\right)}{2^a \pi^{1+a} e^{a\gamma}}$$

$$= \frac{1}{(2\pi e^{\gamma})^a} \frac{\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)}, \quad 0(a)2,$$

where $\{x\}$ has the meaning assigned to it in the last section, and $[x]$ has the same meaning as in §§ 17–19.*

§ 72. It can be shown by general reasoning, or by means of the sine-series, that $[x] = \{1 - x\}$, where, if $1 - x$ is negative, $\{1 - x\}$ is to be defined as $-\{x - 1\}$.

Thus, the exponent of 2 in the above formula may be expressed in the form $\frac{1}{2}\{1 - 2a\} + \frac{1}{4}\{1 - 4a\} + \frac{1}{8}\{1 - 8a\} + \&c.$ The form given in the last section seems, however, to be preferable.

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* In the definition of $[x]$ in § 19 (p. 281), $x - n$ is printed instead of $n - x$. The definition in § 17 (p. 280) is correct.

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It will be noticed that the theorem may be deduced from that of § 19 by replacing a by $1 - a$, since

$$[n(1-a)] = \{na\}, \quad [m(1-a)] = -[ma],$$

if n be uneven, and m be even.

§ 73. As an example of the application of the general theorem in § 71, let $a = \frac{1}{7}$. We find

$$\{(14n+1)\tfrac{1}{7}\} = \tfrac{1}{7}, \quad \{(14n+9)\tfrac{1}{7}\} = -\tfrac{5}{7},$$

$$\{(14n+3)\tfrac{1}{7}\} = \tfrac{3}{7}, \quad \{(14n+11)\tfrac{1}{7}\} = -\tfrac{3}{7},$$

$$\{(14n+5)\tfrac{1}{7}\} = \tfrac{5}{7}, \quad \{(14n+13)\tfrac{1}{7}\} = -\tfrac{1}{7},$$

and, by proceeding as in §§ 52 and 53 and using the cycles there given, we see that, if p be of the form $14n + 1$, the exponent

$$\frac{1}{p}\{pa\} + \frac{1}{p^2}\{p^2a\} + \frac{1}{p^3}\{p^3a\} + \&c.$$

is equal to $\frac{1}{p-1}$; if p be of the form $14n + 3$, the exponent

$$= \frac{3}{7}\frac{1}{p} - \frac{5}{7}\frac{1}{p^2} - \frac{1}{7}\frac{1}{p^3} - \frac{3}{7}\frac{1}{p^4} + \frac{5}{7}\frac{1}{p^5} + \frac{1}{7}\frac{1}{p^6} + \&c.$$

$$= \frac{1}{7}\frac{3p^2-5p-1}{p^3+1},$$

and so on. The exponent of 2 is the same as in § 54, viz. $\frac{25}{49}$. The theorem thus gives the formula

$$(14n+1)^{\frac{1}{p-1}}(14n+3)^{\frac{3p^2-5p-1}{p^3+1}}(14n+5)^{\frac{5p^2-3p-1}{p^3+1}}$$

$$\times (14n+9)^{\frac{-5p^2-3p+1}{p^3-1}}(14n+11)^{\frac{-3p^2-5p+1}{p^3-1}}(14n+13)^{-\frac{1}{p+1}}$$

$$= \frac{2^{\frac{1}{7}p}(\cos\frac{1}{14}\pi)^7\,\Gamma^{14}(\frac{3}{7})}{\pi^8 e^7} = \frac{2^{\frac{1}{7}p}\,\Gamma^7(\frac{3}{7})}{\pi e^7\,\Gamma^7(\frac{4}{7})},$$

agreeing with the result in § 54, which was obtained by putting $a = \frac{3}{7}$ in the general formula of § 19.

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Second general theorem, §§ 74–76.

§ 74. By multiplying the general formulæ of §§ 19 and 71, we find

$$3^{\frac{1}{3}[3a]+\frac{1}{3}\{3a\}+\frac{1}{9}[9a]+\frac{1}{9}\{9a\}+\cdots}5^{\frac{1}{5}[5a]+\frac{1}{5}\{5a\}+\frac{1}{25}[25a]+\frac{1}{25}\{25a\}+\cdots}\cdots$$

$$=\frac{\sin a\pi\,\Gamma^2\left(\frac{a}{2}\right)\Gamma^2\left(\frac{1-a}{2}\right)}{4\pi^3 e^\gamma}=\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1-a}{2}\right)}{2\pi e^\gamma\,\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(1-\frac{a}{2}\right)},$$

a being, as before, intermediate to 0 and 2. By the multiplication, the prime number 2 has disappeared from the product.

Now, x being any positive quantity,

$$[x]+\{x\}=n-x+x-m=n-m,$$

where n and m are respectively the nearest uneven integer, and the nearest even integer, to x.

If the integer next below x is even, $n-m$ is equal to $+1$, and if the integer next below x is uneven, it is equal to -1.

Using therefore the symbol $|x|$ to denote $+1$ or -1, according as the greatest integer contained in x (*i.e.*, the integer next inferior to x) is even or uneven, we may express the theorem in the form

$$3^{\frac{1}{3}|3a|+\frac{1}{9}|9a|+\cdots}5^{\frac{1}{5}|5a|+\frac{1}{25}|25a|+\cdots}7^{\frac{1}{7}|7a|+\frac{1}{49}|49a|+\cdots}\cdots$$

$$=\frac{\sin a\pi\,\Gamma^2\left(\frac{a}{2}\right)\Gamma^2\left(\frac{1-a}{2}\right)}{4\pi^3 e^\gamma}=\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1-a}{2}\right)}{2\pi e^\gamma\,\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(1-\frac{a}{2}\right)},\quad 0(a)2.$$

When x is an integer, the value of $|x|$ is supposed to be zero.

§ 75. Since $|x|$ can only have the values ± 1 when x is fractional, the exponent of a prime number p must be (if not zero) of the form

$$\pm\frac{1}{p}\pm\frac{1}{p^2}\pm\frac{1}{p^3}\pm\frac{1}{p^4}+\&c.$$

It is evident therefore that, if the denominator of a be q,

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prime number q, the exponent of a prime of the form $2qn+r$, r being prime to $2q$, must be of the form

$$\frac{1}{q}\,\frac{\pm p^{q-1}\pm p^{q-2}\pm\ldots+p\pm 1}{p^q\pm 1}.$$

Of course, in this expression there may be factors common to the numerator and denominator (as occur in the case of primes of the forms $2qn+1$ and $2qn+q-1$). Results of the same kind clearly hold good in the case when q is not a prime, if $a=\frac{m}{q}$, m being prime to q.

§ 76. The formula obtained in § 74 may be readily deduced by a direct method from the original theorem of § 1. For, by multiplying together the first formula in § 18, and the first formula in § 69 (which was derived from that in § 18 by substituting $1-a$ for a), we find

$$\left\{\frac{\sin a\pi\ \Gamma^2\left(\frac{a}{2}\right)\Gamma^2\left(\frac{1-a}{2}\right)}{4\pi^2 e^\gamma}\right\}^{\frac{1}{4}\pi} = 3^{\frac{\sin 3a\pi}{3}}\,5^{-\frac{\sin 5a\pi}{5}}\,7^{\frac{\sin 7a\pi}{7}}\ldots.$$

In this product, when all the numbers are resolved into their prime factors, the exponent of 3 is

$$\frac{\sin 3a\pi}{3}+2\frac{\sin 9a\pi}{9}+\frac{\sin 15a\pi}{15}+\&c.;$$

that of 5 is

$$\frac{\sin 5a\pi}{5}+\frac{\sin 15a\pi}{15}+2\frac{\sin 25a\pi}{25}+\&c.,$$

and so on.

Now, proceeding as in § 18, we find that, if

$$\chi(x)=\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\frac{\sin 7x}{7}+\&c.,$$

then

$$\begin{aligned}\chi(x) &= \tfrac{1}{4}\pi, \quad 0(x)\ \pi,\\ &= -\tfrac{1}{4}\pi, \quad \pi(x)\ 2\pi,\\ &= \tfrac{1}{4}\pi, \quad 2\pi(x)\ 3\pi,\end{aligned}$$

and so on.

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Thus

$$\sin \pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{\sin 7\pi x}{7} + \&c. = \tfrac{1}{4}\pi \, |x| ,$$

where, as in § 74, $|x|$ denotes $+1$ or -1 according as the integer next inferior to x is even or uneven, and is equal to zero if x is an integer.

Thus extracting the $\left(\frac{\pi}{4}\right)^{\text{th}}$ root of each side of the equation, we find

$$\frac{\sin a\pi \, \Gamma^2\left(\frac{a}{2}\right) \Gamma^2\left(\frac{1-a}{2}\right)}{4\pi^3 e^{\gamma}} = 3^{\frac{1}{3}|3a|+\frac{1}{9}|9a|+\cdots} 5^{\frac{1}{5}|5a|+\frac{1}{25}|25a|+\cdots} \ldots, 0(a)2,$$

as in § 74.

Application of the new theorem to the case $q = 7$, §§ 77–79.

§ 77. Applying the theorem of § 74 to the cases $a = \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$, we form at once the following table:—

	$a = \frac{1}{7}$	$a = \frac{2}{7}$	$a = \frac{3}{7}$	Cycles.
$\lvert (14n+1)a \rvert$	$+1$	$+1$	$+1$	1 ,
$\lvert (14n+3)a \rvert$	$+1$	$+1$	-1	3 , 9 , 13, 11, 5, 1
$\lvert (14n+5)a \rvert$	$+1$	-1	$+1$	5 , 11, 13, 9 , 3, 1
$\lvert (14n+9)a \rvert$	-1	$+1$	-1	9 , 11, 1
$\lvert (14n+11)a \rvert$	-1	-1	$+1$	11, 9 , 1
$\lvert (14n+13)a \rvert$	-1	-1	-1	13, 1

The cycles for the powers of the primes of the forms $14n+1$, $14n+3$, &c. are shown in the last column. These cycles were given in § 52.

It is interesting to notice how very easily the above table is formed. In order to obtain the first three columns it is

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only necessary to determine whether the integral part of ra, for the six values of r and the three values of a, is even or uneven.

§ 78. By mere inspection of the above table we may write down at once the three products, which the general formula gives, corresponding to the three values of a.

These products are:—

	$a=\frac{1}{7}$,	$a=\frac{2}{7}$,	$a=\frac{3}{7}$,
$p=14n+1$, exp. =	$\frac{1}{p-1}$,	$\frac{1}{p-1}$,	$\frac{1}{p-1}$,
$p=14n+3$, exp. =	$\frac{p^2-p-1}{p^3+1}$,	$\frac{p^2+p-1}{p^3+1}$,	$\frac{-p^2-p-1}{p^3+1}$,
$p=14n+5$, exp. =	$\frac{p^2-p-1}{p^3+1}$,	$\frac{-p^2-p-1}{p^3+1}$,	$\frac{p^2+p-1}{p^3+1}$,
$p=14n+9$, exp. =	$\frac{-p^2-p+1}{p^3-1}$,	$\frac{p^2-p+1}{p^3-1}$,	$\frac{-p^2+p+1}{p^3-1}$,
$p=14n+11$, exp. =	$\frac{-p^2-p+1}{p^3-1}$,	$\frac{-p^2+p+1}{p^3-1}$,	$\frac{p^2-p+1}{p^3-1}$,
$p=14n+13$, exp. =	$-\frac{1}{p+1}$,	$-\frac{1}{p+1}$,	$-\frac{1}{p+1}$.

These products are the same as (A), (B), (C) of § 55, as they evidently should be, since the formulæ given by the new general theorm must be equivalent to the results obtained by multiplying together the pairs of formulæ, given by the theorem of § 19, for complementary values of a (*i.e.* pairs of values of a whose sum is unity). The values of the products given by the theorem are exactly the same as in the formulæ (A), (B), (C).

§ 79. If we assume that only half of the $q-1$ formulæ obtained from the general theorem of § 19 for any prime value of q are independent (as is probably true, but requires proof), then the theorem of § 74 gives all the independent

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results. The process is very simple and the $\frac{1}{2}(q-1)$ results are expressed in a very convenient form.

Application of the new theorem to $q=11$, §§ 80–89.

§ 80. As an application of the theorem in § 74 to another value of q, we may take the case of $q=11$.

We first form the following table of the values of $|(22n+r)a|$ for the different values of r and a:—

a	Π^{1}	Π^{2}	Π^{3}	Π^{4}	Π^{5}
$\|(22n+1)a\|$	+1	+1	+1	+1	+1
$\|(22n+3)a\|$	+1	+1	+1	−1	−1
$\|(22n+5)a\|$	+1	+1	−1	−1	+1
$\|(22n+7)a\|$	+1	−1	−1	+1	−1
$\|(22n+9)a\|$	+1	−1	+1	−1	+1
$\|(22n+13)a\|$	−1	+1	−1	+1	−1
$\|(22n+15)a\|$	−1	+1	+1	−1	+1
$\|(22n+17)a\|$	−1	−1	+1	+1	−1
$\|(22n+19)a\|$	−1	−1	−1	+1	+1
$\|(22n+21)a\|$	−1	−1	−1	−1	−1

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The cycles of the residues, with respect to the modulus 22, of the first, second, third, &c. powers of numbers of the forms $22n+1$, $22n+3$, &c., are easily found to be:—

$22n+1$; 1;

$22n+3$; 3, 9, 5, 15, 1;

$22n+5$; 5, 3, 15, 9, 1;

$22n+7$; 7, 5, 13, 3, 21, 15, 17, 9, 19, 1;

$22n+9$; 9, 15, 3, 5, 1;

$22n+13$; 13, 15, 19, 5, 21, 9, 7, 3, 17, 1;

$22n+15$; 15, 5, 9, 3, 1;

$22n+17$; 17, 3, 7, 9, 21, 5, 19, 15, 13, 1;

$22n+19$; 19, 9, 17, 15, 21, 3, 13, 5, 7, 1;

$22n+21$; 21, 1.

§ 81. It is evident that, for all the values of a, the exponent of p is $\frac{1}{p-1}$ if p is of the form $22n+1$, and is $-\frac{1}{p+1}$ if p is of the form $-\frac{1}{p+1}$.

The exponents which correspond to the five values of a, when p is of the other eight forms, may be conveniently exhibited as follows:

Form of prime.	Numerator of exponent.	Denominator of exponent.
$22n+3$,	$p^4+p^3+p^2-p+1$,	p^5-1,
	$p^4-p^3+p^2+p+1$,	
	$p^4+p^3-p^2+p+1$,	
	$-p^4-p^3-p^2-p+1$,	
	$-p^4+p^3+p^2+p+1$,	
$22n+5$,	$p^4+p^3-p^2+p+1$,	p^5-1,
	$p^4+p^3+p^2-p+1$,	
	$-p^4+p^3+p^2+p+1$,	
	$-p^4-p^3-p^2-p+1$,	
	$p^4-p^3+p^2+p+1$,	

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Form of prime	Numerator of exponent	Denominator of exponent
$22n+7$,	$p^6+p^5-p^3+p-1$,	p^3+1,
	$-p^6+p^5+p^4+p-1$,	
	$-p^6-p^5-p^4+p-1$,	
	$p^6-p^5+p^3-p-1$,	
	$-p^6+p^5-p^3-p-1$,	
$22n+9$,	$p^6-p^5+p^3+p+1$,	p^3-1,
	$-p^6+p^5+p^3+p+1$,	
	$p^6+p^5+p^3-p+1$,	
	$-p^6-p^5-p^3-p+1$,	
	$p^6+p^5-p^3+p+1$,	
$22n+13$,	$-p^6-p^5-p^3+p-1$,	p^3+1,
	$p^6+p^5-p^3+p-1$,	
	$-p^6+p^5-p^3-p-1$,	
	$p^6-p^5+p^3-p-1$,	
	$-p^6+p^5+p^3+p-1$,	
$22n+15$,	$-p^6+p^5+p^3+p+1$,	p^3-1,
	$p^6+p^5-p^3+p+1$,	
	$p^6-p^5+p^3+p+1$,	
	$-p^6-p^5-p^3-p+1$,	
	$p^6+p^5+p^3-p+1$,	
$22n+17$,	$-p^6+p^5+p^3+p-1$,	p^3+1,
	$-p^6+p^5-p^3-p-1$,	
	$p^6+p^5-p^3+p-1$,	
	$p^6-p^5+p^3-p-1$,	
	$-p^6-p^5-p^3+p-1$,	

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Form of prime.	Numerator of exponent.	Denominator of exponent.
$22n+19$,	$-p^4+p^3-p^2-p-1$,	p^5+1,
	$-p^4-p^3-p^2+p-1$,	
	$-p^4+p^3+p^2+p-1$,	
	$p^4-p^3+p^2-p-1$,	
	$p^4+p^3-p^2+p-1$.	

In the above list the five values of the numerator of the exponent of the prime p, which correspond to the five values $a = {}_{11}^{1}, {}_{11}^{2}, {}_{11}^{3}, {}_{11}^{4}, {}_{11}^{5}$ respectively, are written under one another in this order. The denominator is the same for all five numerators.

§ 82. The construction of the list is extremely simple. The group of numerators corresponding to each form of the prime p is most easily written down in columns. Thus, taking the case of $p = 22n+3$, the cycle of residues is 3, 9, 5, 15, 1. The signs in the p^4 column are those in the table of § 80 which are opposite $|(22n+3)a|$, viz. +, +, +, −, −; the signs in the p^3 column are those which are opposite to $|(22n+3)a|$, viz. +, +, +, −, −, and so on. It is only necessary to use the first five residues in the cycle. The denominator is p^5-1 or p^5+1 according as the cycle contains five or ten residues.

§ 83. The values of the five products are (taking only the first form given by the theorem of § 74) respectively

$$\text{(i)}\quad \frac{\sin {}_{11}^{1}\pi\,\Gamma^3(\frac{1}{22})\,\Gamma^3(\frac{10}{22})}{4\pi^3 e^{\gamma}} = V_1, \text{ say}$$

$$\text{(ii)}\quad \frac{\sin {}_{11}^{2}\pi\,\Gamma^3(\frac{2}{22})\,\Gamma^3(\frac{9}{22})}{4\pi^3 e^{\gamma}} = V_2, \text{ ,,}$$

$$\text{(iii)}\quad \frac{\sin {}_{11}^{3}\pi\,\Gamma^3(\frac{3}{22})\,\Gamma^3(\frac{8}{22})}{4\pi^3 e^{\gamma}} = V_3, \text{ ,,}$$

$$\text{(iv)}\quad \frac{\sin {}_{11}^{4}\pi\,\Gamma^3(\frac{4}{22})\,\Gamma^3(\frac{7}{22})}{4\pi^3 e^{\gamma}} = V_4, \text{ ,,}$$

$$\text{(v)}\quad \frac{\sin {}_{11}^{5}\pi\,\Gamma^3(\frac{5}{22})\,\Gamma^3(\frac{6}{22})}{4\pi^3 e^{\gamma}} = V_5, \text{ ,,}$$

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Thus, the first formula written at length is

$$(22n+1)^{\frac{1}{p-1}}(22n+3)^{\frac{p^4+p^3+p^2-p+1}{p^5-1}}(22n+5)^{\frac{p^4+p^3-p^2+p+1}{p^5-1}}$$

$$\times\,(22n+7)^{\frac{p^4+p^3-p^2+p-1}{p^5+1}}(22n+9)^{\frac{p^4-p^3+p^2+p+1}{p^5-1}}$$

$$\times\,(22n+13)^{\frac{-p^4-p^3-p^2+p-1}{p^5+1}}(22n+15)^{\frac{-p^4+p^3+p^2+p+1}{p^5-1}}$$

$$\times\,(22n+17)^{\frac{-p^4+p^3+p^2+p-1}{p^5+1}}(22n+19)^{\frac{-p^4+p^3-p^2-p-1}{p^5+1}}$$

$$\times\,(22n+21)^{-\frac{1}{p+1}} = V_1 = \frac{\sin\frac{1}{11}\pi\,\Gamma^2(\frac{1}{22})\,\Gamma^2(\frac{10}{22})}{4\pi^3 e^\gamma}.$$

§ 84. It will be noticed that the fourth product differs from the others in having several exponents alike. In this respect it resembles the product (A) of § 55 which was derived from the cases $a=\frac{1}{7}$ and $\frac{6}{7}$. The exponents in this product are

p	numerator.	denominator.
$22n+1$,	1,	$p-1$,
$22n+3$ $22n+5$ } ,	$-p^4-p^3-p^2-p+1$,	p^5-1,
$22n+7$,	$p^4-p^3+p^2-p-1$,	p^5+1,
$22n+9$,	$-p^4-p^3-p^2-p+1$,	p^5-1,
$22n+13$,	$p^4-p^3+p^2-p-1$,	p^5+1,
$22n+15$,	$-p^4-p^3-p^2-p+1$,	p^5-1,
$22n+17$ $22n+19$ } ,	$p^4-p^3+p^2-p-1$,	p^5+1,
$22n+21$,	-1,	$p+1$,

and since

$$\frac{-p^4-p^3-p^2-p+1}{p^5-1} = -\frac{1}{p-1}+\frac{2}{p^5-1},$$

and

$$\frac{p^4-p^3+p^2-p-1}{p^5+1} = \frac{1}{p+1}-\frac{2}{p^5+1},$$

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we may express the result in the form

$$(22n+1)^{\frac{1}{p-1}}(22n+3,5,9,15)^{-\frac{1}{p-1}+\frac{2}{p^2-1}}$$
$$\times (22n+7,13,17,19)^{\frac{1}{p+1}-\frac{2}{p^2+1}}(22n+21)^{-\frac{1}{p+1}}$$
$$= V_4 = \frac{\sin\frac{4}{11}\pi\,\Gamma^2(\frac{4}{22})\,\Gamma^2(\frac{7}{22})}{4\pi^3 e^\gamma}.$$

§ 85. By adding together the first, second, third and fifth exponents in § 81 (*i.e.*, adding together all except the fourth), corresponding to each form of p, and dividing the sums so formed by 2, we obtain the following product:

p	numerator.	denominator.
$22n+1$,	2,	$p-1$,
$22n+3$, $22n+5$,	$p^4+p^3+p^2+p+2$,	p^5-1,
$22n+7$,	$-p^4+p^3-p^2+p-2$,	p^5+1,
$22n+9$,	$p^4+p^3+p^2+p+2$,	p^5-1,
$22n+13$,	$-p^4+p^3-p^2+p-2$,	p^5+1,
$22n+15$,	$p^4+p^3+p^2+p+2$,	p^5-1,
$22n+17$, $22n+19$,	$-p^4+p^3-p^2+p-2$,	p^5+1,
$22n+21$,	-2,	$p+1$.

Since

$$\frac{p^4+p^3+p^2+p+2}{p^5-1} = \frac{1}{p-1}+\frac{1}{p^5-1},$$

and

$$\frac{-p^4+p^3-p^2+p-2}{p^5+1} = -\frac{1}{p+1}-\frac{1}{p^5+1},$$

we may express the result in the form

$$(22n+1)^{\frac{2}{p-1}}(22n+3,5,9,15)^{\frac{1}{p-1}+\frac{1}{p^2-1}}$$
$$\times (22n+7,13,17,19)^{-\frac{1}{p-1}-\frac{1}{p^2+1}}(22n+21)^{-\frac{2}{p+1}}$$
$$= (V_1V_2V_3V_4)^{\frac{1}{2}} = \frac{(\sin\frac{1}{11}\pi\sin\frac{2}{11}\pi\sin\frac{3}{11}\pi\sin\frac{5}{11}\pi)^{\frac{1}{2}}}{2^4\pi^6 e^{2\gamma}}$$
$$\times \Gamma(\tfrac{1}{22})\,\Gamma(\tfrac{2}{22})\,\Gamma(\tfrac{3}{22})\,\Gamma(\tfrac{5}{22})\,\Gamma(\tfrac{6}{22})\,\Gamma(\tfrac{8}{22})\,\Gamma(\tfrac{9}{22})\,\Gamma(\tfrac{10}{22}).$$

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In this product the primes of the different forms are grouped together in exactly the same manner as in the product obtained in the preceding section.

§ 86. By squaring the second product and dividing by the first, and finally extracting the cube root, we find

$$(22n+1, 3, 5, 9, 15)^{\frac{1}{p-1}}(22n+7, 13, 17, 19, 21)^{-\frac{1}{p-1}} = \left(\frac{V_1V_2V_3V_5}{V_4}\right)^{\frac{1}{3}} \ldots(\alpha).$$

Also, by multiplying together the two products as they stand and extracting the cube root, we find

$$(22n+1)^{\frac{1}{p-1}}(22n+3, 5, 9, 15)^{\frac{1}{p-1}} \times (22n+7, 13, 17, 19)^{-\frac{1}{p+1}}(22n+21)^{-\frac{1}{p-1}} = (V_1V_2V_3V_4^{-1}V_5)^{\frac{1}{3}} \ldots(\beta).$$

§ 87. By adding together the first and fourth exponents in the list in § 81, and also the second and fourth, the third and fourth, and the fifth and fourth, we obtain four products, the exponents of which, when divided by 2 are as follows:

p	numerators				denominator
$22n+1$,	1,	1,	1,	1,	$p-1$,
$22n+3$,	$-p+1$,	$-p^3+1$,	$-p^2+1$,	$-p^4+1$,	p^5-1,
$22n+5$,	$-p^3+1$,	$-p+1$,	$-p^4+1$,	$-p^3+1$,	p^5-1,
$22n+7$,	p^4-1,	p^2-1,	$-p^3-1$,	$-p-1$,	p^5+1,
$22n+9$,	$-p^3+1$,	$-p^4+1$,	$-p+1$,	$-p^2+1$,	p^5-1,
$22n+13$,	$-p^3-1$,	p^4-1,	$-p-1$,	p^2-1,	p^5+1,
$22n+15$,	$-p^4+1$,	$-p^2+1$,	$-p^3+1$,	$-p+1$,	p^5-1,
$22n+17$,	p^2-1,	$-p-1$,	p^4-1,	$-p^3-1$,	p^5+1,
$22n+19$,	$-p-1$,	$-p^3-1$,	p^2-1,	p^4-1,	p^5+1,
$22n+21$,	-1,	-1,	-1,	-1,	$p+1$.

The values of these four products are respectively

$$(V_1V_4)^{\frac{1}{2}},\ (V_2V_4)^{\frac{1}{2}},\ (V_3V_4)^{\frac{1}{2}},\ (V_4V_5)^{\frac{1}{2}}.$$

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§ 88. By dividing each of these four products by the product (β) of § 86 we obtain the four following formulæ, in which the primes of the forms $22n+1$ and $22n+21$ do not occur, and the numerators of the exponents each consist of a single term:

$$(22n+3)^{-\frac{p}{p^5-1}}(22n+5)^{-\frac{p^2}{p^5-1}}(22n+7)^{\frac{p^4}{p^5+1}}$$
$$\times (22n+9)^{-\frac{p^3}{p^5-1}}(22n+13)^{-\frac{p^3}{p^5+1}}(22n+15)^{-\frac{p^4}{p^5-1}}$$
$$\times (22n+17)^{\frac{p^2}{p^5+1}}(22n+19)^{-\frac{p}{p^5+1}} = V_1^{\frac{1}{2}}V_2^{-\frac{1}{2}}V_3^{-\frac{1}{2}}V_4^{\frac{1}{2}}V_5^{-\frac{1}{2}} \quad (\gamma),$$

$$(22n+3)^{-\frac{p^3}{p^5-1}}(22n+5)^{-\frac{p}{p^5-1}}(22n+7)^{\frac{p^2}{p^5+1}}$$
$$\times (22n+9)^{-\frac{p^4}{p^5-1}}(22n+13)^{\frac{p^4}{p^5+1}}(22n+15)^{-\frac{p^2}{p^5-1}}$$
$$\times (22n+17)^{-\frac{p}{p^5+1}}(22n+19)^{-\frac{p^3}{p^5+1}} = V_1^{-\frac{1}{2}}V_2^{\frac{1}{2}}V_3^{-\frac{1}{2}}V_4^{\frac{1}{2}}V_5^{-\frac{1}{2}} \quad (\delta),$$

$$(22n+3)^{-\frac{p^3}{p^5-1}}(22n+5)^{-\frac{p^4}{p^5-1}}(22n+7)^{-\frac{p^3}{p^5+1}}$$
$$\times (22n+9)^{-\frac{p}{p^5-1}}(22n+13)^{-\frac{p}{p^5+1}}(22n+15)^{-\frac{p^3}{p^5-1}}$$
$$\times (22n+17)^{\frac{p^4}{p^5+1}}(22n+19)^{\frac{p^2}{p^5+1}} = V_1^{-\frac{1}{2}}V_2^{-\frac{1}{2}}V_3^{\frac{1}{2}}V_4^{\frac{1}{2}}V_5^{-\frac{1}{2}} \quad \ldots(\epsilon),$$

$$(22n+3)^{-\frac{p^4}{p^5-1}}(22n+5)^{-\frac{p^3}{p^5-1}}(22n+7)^{-\frac{p}{p^5+1}}$$
$$\times (22n+9)^{-\frac{p^2}{p^5-1}}(22n+13)^{\frac{p^2}{p^5+1}}(22n+15)^{-\frac{p}{p^5-1}}$$
$$\times (22n+17)^{-\frac{p^3}{p^5+1}}(22n+19)^{\frac{p^4}{p^5+1}} = V_1^{-\frac{1}{2}}V_2^{-\frac{1}{2}}V_3^{-\frac{1}{2}}V_4^{\frac{1}{2}}V_5^{\frac{1}{2}} \quad (\zeta).$$

The last four formulæ (γ), (δ), (ϵ), (ζ), together with either (α) or (β) of § 86, contain probably the simplest group of products, which are derivable from the system of results given by the general theorem.

§ 89. Other systems of interest presenting a certain amount of regularity may be deduced from the fundamental results of § 81. Thus, for example, by combining by addition

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and subtraction every corresponding pair of exponents, for the different forms of the primes, we obtain a system of twenty formulæ, in ten of which the numerators of the exponents (exclusive of the exponents of the primes of the forms $22n+1$ and $22n+23$) are binomial expressions, and in ten trinomial expressions, *i.e.* they are of the forms

$$\frac{\pm p^m \pm p^n}{p^5 \pm 1} \text{ or } \frac{\pm p^m \pm p^n \pm p^r}{p^5 \pm 1},$$

where m, n, r have the values 0, 1, 2, 3, 4. In the formulæ obtained by subtraction the primes of the forms $22n+1$ and $22n+3$ do not occur, and in those obtained by addition, the exponents of these primes are $\frac{1}{p-1}$ and $-\frac{1}{p+1}$ respectively.

Completion of the list in § 65, §§ 90, 91.

§ 90. The list of formulæ in § 65 may be completed by the addition of those obtained for the case $q=11$. Expressed in the series form these latter may be written as follows:

(xiv) If $\quad A_p = \frac{1}{p-1}$, or $-\frac{1}{p+1}$,

according as

$$p \equiv 1, 3, 5, 9, 15, \text{ or } 7, 13, 17, 19, 21, \text{ mod. } 22,$$

then $$\Sigma_1^\infty A_p \log p = \tfrac{1}{8} \log \left(\frac{V_1 V_2 V_3 V_5}{V_4} \right).$$

(xv) If $A_p = \frac{1}{p-1}$, $\frac{1}{p^5-1}$, $-\frac{1}{p^5+1}$, $-\frac{1}{p+1}$,

according as

$$p \equiv 1, \text{ or } 3, 5, 9, 15, \text{ or } 7, 13\ 17, 19, \text{ or } 21, \text{ mod. } 22,$$

then $$\Sigma_1^\infty A_p \log p = \tfrac{1}{8} \log (V_1 V_2 V_3 V_4^2 V_5).$$

(xvi) If

$$A_p = -\frac{p}{p^5-1}, -\frac{p^2}{p^5-1}, \frac{p^4}{p^5+1}, -\frac{p^3}{p^5-1},$$
$$-\frac{p^3}{p^5+1}, -\frac{p^4}{p^5-1}, \frac{p^2}{p^5+1}, -\frac{p}{p^5+1},$$

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according as

$p \equiv 3$, or 5, or 7, or 9, or 13, or 15, or 17, or 19, mod. 22,

then $\Sigma_1^\infty A_p \log p = \frac{1}{8} \log\left(\frac{V_1^2 V_4}{V_2 V_3 V_5}\right)$.

(xvii) If

$$A_p = -\frac{p^3}{p^5-1}, \; -\frac{p}{p^5-1}, \; \frac{p^2}{p^5+1}, \; -\frac{p^4}{p^5-1},$$
$$\frac{p^4}{p^5+1}, \; -\frac{p^2}{p^5-1}, \; -\frac{p}{p^5+1}, \; -\frac{p^3}{p^5+1},$$

according as

$p \equiv 3$, or 5, or 7, or 9, or 13, or 15, or 17, or 19, mod. 22,

then $\Sigma_1^\infty A_p \log p = \frac{1}{8} \log\left(\frac{V_2^2 V_4}{V_1 V_3 V_5}\right)$.

(xviii) If

$$A_p = -\frac{p^2}{p^5-1}, \; -\frac{p^4}{p^5-1}, \; -\frac{p^3}{p^5+1}, \; -\frac{p}{p^5-1},$$
$$-\frac{p}{p^5+1}, \; -\frac{p^3}{p^5-1}, \; \frac{p^4}{p^5+1}, \; \frac{p^2}{p^5+1},$$

according as

$p \equiv 3$, or 5, or 7, or 9, or 13, or 15, or 17, or 19, mod. 22,

then $\Sigma_1^\infty A_p \log p = \frac{1}{8} \log\left(\frac{V_3^2 V_4}{V_1 V_2 V_5}\right)$.

(xix) If

$$A_p = -\frac{p^4}{p^5-1}, \; -\frac{p^2}{p^5-1}, \; -\frac{p}{p^5+1}, \; -\frac{p^3}{p^5-1},$$
$$\frac{p^2}{p^5+1}, \; -\frac{p}{p^5-1}, \; -\frac{p^3}{p^5+1}, \; \frac{p^4}{p^5+1},$$

according as

$p \equiv 3$, or 5, or 7, or 9, or 13, or 15, or 17, or 19, mod. 22,

then $\Sigma_1 A_p \log p = \frac{1}{8} \log\left(\frac{V_4 V_5^2}{V_1 V_2 V_3}\right)$.

The values of V_1, V_2, V_3, V_4, V_5 were given in § 83.

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§ 91. Of all the products for which values have been obtained in this paper perhaps the most interesting are those in which the only exponents which occur are those of the forms $\frac{1}{p-1}$ and $-\frac{1}{p+1}$.

The following is the list of these products:—

$$(4n+1)^{\frac{1}{p-1}}(4n+3)^{-\frac{1}{p-1}} \qquad (\S\,21, \text{p. } 282),$$

$$(6n+1)^{\frac{1}{p-1}}(6n+5)^{-\frac{1}{p-1}} \qquad (\S\,22, \text{p. } 283),$$

$$(8n+1, 3)^{\frac{1}{p-1}}(8n+5, 7)^{-\frac{1}{p-1}} \qquad (\S\,29, \text{p. } 289),$$

$$(14n+1, 9, 11)^{\frac{1}{p-1}}(14n+3, 5, 13)^{-\frac{1}{p-1}} \qquad (\S\,60, \text{p. } 309),$$

$$(22n+1, 3, 5, 9, 15)^{\frac{1}{p-1}}(22n+7, 13, 17, 19, 21)^{-\frac{1}{p+1}} (\S\,86, \text{p. } 330).$$

The references give the sections and pages where the values of the products were obtained. The products

$$(8n+3)^{\frac{p}{p^2-1}}(8n+5)^{-\frac{p}{p^2-1}} \quad (\S\,30, \text{p. } 290),$$

$$(10n+3)^{\frac{p}{p^2+1}}(10n+7)^{-\frac{p}{p^2+1}} \quad (\S\,47, \text{p. } 300),$$

$$(12n+5)^{\frac{p}{p^2-1}}(12n+7)^{-\frac{p}{p^2-1}} \quad (\S\,35, \text{p. } 293),$$

are also noticeable on account of their simplicity.

The values of all the products are given in a convenient form for reference in the lists of §§ 65 and 90.

Product-formulæ for the Gamma function, § 92.

§ 92. In this paper the formulæ have been regarded as giving the values of certain products depending upon prime numbers, these values being expressed in terms of Gamma functions, and in some cases in terms of elliptic integrals also. But we may regard the general formulæ also as affording product-values for the Gamma functions. By simply writing Kummer's series for $\log\Gamma(x)$, viz.*

$$\log\Gamma(x) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\log\sin\pi x + \tfrac{1}{2}\{\gamma + \log(2\pi)\}(1-2x)$$
$$= \frac{1}{\pi}\left\{\frac{\log 2}{2}\sin 4\pi x + \frac{\log 3}{3}\sin 6\pi x + \&c.\right\}, 0(x)1,$$

* *Crelle's Journal*, Vol. xxiv., p. 4.

in the form of a product, M. Rogel obtained the formula

$$\frac{(\sin \pi x)^{\frac{1}{2}} \Gamma(x)}{(2e^{\gamma})^{\frac{1}{2}-x} \pi^{1-x}} = \Pi_2^{\infty} n^{\frac{\sin 2n\pi x}{n\pi}}, \quad 0(x)1,$$

by which the Gamma function is expressed as a product of all the natural numbers, each being raised to an exponent depending upon the number and involving a circular function; and, by transforming this product as in § 18, we obtain the theorem (§ 19)

$$\frac{(\sin \pi x)^{\frac{1}{2}} \Gamma(x)}{(2e^{\gamma})^{\frac{1}{2}-x} \pi^{1-x}} = \Pi_2^{\infty} p^{\frac{1}{2p}[2px] + \frac{1}{2p^2}[2p^2x] + \frac{1}{2p^3}[2p^3x] + \ldots}, \quad 0(x)1,$$

by which the Gamma function is expressed as a product of prime numbers only, each prime p being raised to an exponent which is a rational algebraical function of p.

It is to be observed that in reality the products represent a quotient of Gamma functions, *i.e.*, the formulæ are

$$\frac{\Gamma(x)}{\Gamma(1-x)} = (2\pi e^{\gamma})^{1-2x} \Pi_2^{\infty} n^{\frac{2 \sin 2n\pi x}{n\pi}},$$

$$= (2\pi e^{\gamma})^{1-2x} \Pi_2^{\infty} p^{\frac{1}{p}[2px] + \frac{1}{p^2}[2p^2x] + \frac{1}{p^3}[2p^3x] + \ldots}.$$

Expression for the constant A as a product, §§ 93, 94.

§ 93. The constant A, or e^a, may be expressed in terms of a product in which only prime numbers occur by using the method of transformation employed in § 3.

This constant, which occurs in the approximate formula, when n is very large,

$$1^1 . 2^2 . 3^3 \ldots n^n = A n^{\frac{1}{2}n^2 + \frac{1}{4}n + \frac{1}{12}} e^{-\frac{1}{4}n^2},$$

has been recently considered in two papers in the *Messenger*, Vol. XXIII., pp. 145–175, and Vol. XXIV., pp. 1–16. It was shown on p. 3 of the latter paper that

$$\left(\frac{A^{12}}{2\pi e^{\gamma}}\right)^{\frac{1}{\pi^2}} = 1^{\frac{1}{1}} 2^{\frac{1}{2}} 3^{\frac{1}{3}} 4^{\frac{1}{16}} 5^{\frac{1}{15}} \ldots .$$

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Resolving the numbers in the product into their prime factors as in § 3 of the present paper, we see that the exponent of 2

$$= \tfrac{1}{2} + \tfrac{2}{16} + \tfrac{1}{36} + \tfrac{3}{64} + \tfrac{1}{100} + \&c.$$

$$= \frac{1}{2^2}\left\{1 + \frac{2}{2^2} + \frac{1}{3^2} + \frac{3}{4^2} + \&c.\right\}$$

$$= \frac{1}{2^2}\left(1 - \frac{1}{2^2}\right)^{-1}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right) = \frac{1}{2^2 - 1}\,\frac{\pi^2}{6}.$$

Similarly, the exponent of 3 is equal to

$$\frac{1}{3^2 - 1}\,\frac{\pi^2}{6},$$

and so on.

Thus we find

$$\frac{A^{12}}{2\pi e^{\gamma}} = 2^{\frac{1}{2^2-1}} 3^{\frac{1}{3^2-1}} 5^{\frac{1}{5^2-1}} 7^{\frac{1}{7^2-1}} \ldots,$$

in which the product only involves prime numbers.

By taking the logarithm, we may express the result also in the form

$$\Sigma_2^{\infty} \frac{\log p}{p^2 - 1} = 12 \log A - \gamma - \log(2\pi) = 12\alpha - \gamma - \log(2\pi),$$

the summation including all the prime numbers.

§ 94. The general method of transformation employed in § 18, in the case of the sine series, may also be applied to the other trigonometrical series which occur as exponents in the products obtained from the other formulæ in the paper in Vol. XXIV. of the *Messenger*; but I leave the consideration of the results so obtained to a separate paper.

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ERRATA.

p. 280, l. 10 and p. 281, l. 10. *For* $0(a)1$ *read* $0(a)2$. There is no loss of generality in regarding 0 and 1 as the limits of a (for the results are unaltered by substituting $2-a$ for a), but in the statement of the general theorems the limits should not be so restricted. They are given correctly as 0 and 2 in the corresponding theorems in § 69 (p. 317), § 70 (p. 318), and § 74 (p. 320).

p. 281, l. 11. *For* $x-n$ *read* $n-x$. This correction has been made in the note on p. 318.

p. 289, l. 6. 'Will all have exponents of different forms.' It requires to be proved that no two exponents can be alike.

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THE EQUILIBRIUM OF AN ISOTROPIC ELASTIC SOLID ELLIPSOID UNDER THE ACTION OF NORMAL SURFACE FORCES OF THE SECOND DEGREE, AND BODILY FORCES DERIVED FROM A POTENTIAL OF THE SECOND DEGREE.

By C. Chree, M.A.

§ 1. In the *Quarterly Journal* for 1888, pp. 16–20, I showed that my solution* of the elastic solid equations in ascending powers of the coordinates with arbitrary coefficients supplied the means of solving completely the case of an isotropic ellipsoid rotating about a principal axis. It was shown how each of the 9 constants in the expressions for the displacements could be expressed in terms of determinants with 9 rows and columns. Even in a particular case when the elements of the determinants are numerical quantities, the evaluation of the constants would be troublesome, and in the general case my efforts in this direction met with such doubtful success that I laid aside the problem in hopes of a shorter method some day presenting itself. Eventually a method of solution occurred to me which solved much more readily, not merely the original problem, but the more general one presented by the isotropic elastic solid ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

of uniform density ρ, acted on by bodily forces whose components along x, y, z are respectively Px, Qy, Rz, where P, Q, R are constants. This includes the case of mutual gravitation in the elements of the ellipsoid, and that of rotation about a principal axis. It is most simply regarded as the case of bodily forces derived from a potential V of the second degree given by

$$V=\tfrac{1}{2}(Px^2+Qy^2+Rz^2) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

This is the most general form a potential of the second degree can have consistently with the equilibrium of an ellipsoid of three unequal axes.

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* *Quarterly Journal* for 1886, pp. 89–118.

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A year ago I contributed to the Royal Society a paper containing the solution of this problem with some applications, and the results of the investigation without their proof are to appear in the 'Proceedings.' As the method of proof is, I believe, not less novel than the results, and likely to be of at least as much interest to the mathematician, I propose to develope it here.

In addition to the above case of bodily forces I incorporate the further problem, where there act the surface forces*

$$Sx^2 + Ty^2 + Uz^2$$

outwards along the normal at each point of the surface.

This is the most general system of normal forces of the second degree consistent with the equilibrium of an ellipsoid of three unequal axes.

The problem of these surface forces was not considered at all in the paper communicated to the Royal Society.

Notation.

§ 2. The displacements are denoted by α, β, γ, and the dilatation by Δ. For the strains I employ the notation and terminology of Todhunter and Pearson's *History*; viz. s_x, s_y, s_z for the three *stretches*, σ_{yz}, σ_{zx}, σ_{xy} for the three *slides*, where

$$s_x = \frac{d\alpha}{dx}, \ldots, \quad \sigma_{yz} = \frac{d\beta}{dz} + \frac{d\gamma}{dy}, \ldots .$$

In the notation of the same work we have the 6 stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$. The three first I shall term *tractions* or *pressures* according as the sign is positive or negative, the last three *shearing stresses*.

The elastic constants of which use is principally made are "Young's modulus" E, and "Poisson's ratio" η. Occasional use is also make of the "rigidity" n, and "bulk modulus" k. The relations between these 4 constants are

$$\left.\begin{aligned} n &= \tfrac{1}{2}E/(1+\eta) \\ k &= \tfrac{1}{3}E/(1-2\eta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

* A *surface* force Uz^2 is equivalent of course to a force $Uc^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$; so the system Uz^2 could be derived by superposing the uniform traction Uc^2 on the system $-Uc^2\frac{x^2}{a^2} - Uc^2\frac{y^2}{b^2}$.

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The method of solution adopted reverses the ordinary order of procedure, determining the stresses in the first instance, and deducing from their values first those of the strains and finally those of the displacements.

For our present object the most convenient relations between the strains and stresses are thus of the forms

$$\left.\begin{aligned} e_x &= \frac{1}{E}\{\widehat{xx} - \eta(\widehat{yy} + \widehat{zz})\} \\ \sigma_{yz} &= \frac{1}{n}\widehat{yz} = 2(1+\eta)\widehat{yz}/E \end{aligned}\right\} \quad \text{.....(4).}$$

Method of Solution.

§ 3. We commence by assuming the stresses to be of the form

$$\left.\begin{aligned} \widehat{xx} &= A_0 + A_2x^2 + A_2'y^2 + A_2''z^2 \\ \widehat{yy} &= B_0 + B_2x^2 + B_2'y^2 + B_2''z^2 \\ \widehat{zz} &= C_0 + C_2x^2 + C_2'y^2 + C_2''z^2 \\ \widehat{yz} &= 2Lyz \\ \widehat{zx} &= 2Mzx \\ \widehat{xy} &= 2Nxy \end{aligned}\right\} \quad \text{.....(5),}$$

where A_0, A_2, ..., are constants which we have to determine. The assumption may seem wholly arbitrary; but no valid exception can be taken on this ground to the solution obtained by means of it, for the solution satisfies completely *all* the elastic solid equations.

The stresses have to satisfy the three "body-stress" equations

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{zx}}{dz} + \rho Px &= 0 \\ \frac{d\widehat{xy}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz} + \rho Qy &= 0 \\ \frac{d\widehat{zx}}{dx} + \frac{d\widehat{yz}}{dy} + \frac{d\widehat{zz}}{dz} + \rho Rz &= 0 \end{aligned}\right\} \quad \text{.....(6),}$$

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and the three surface equations

$$\left.\begin{aligned} \frac{x}{a^2}\widehat{xx}+\frac{y}{b^2}\widehat{xy}+\frac{z}{c^2}\widehat{zx} &= \frac{x}{a^2}(Sx^2+Ty^2+Uz^2) \\ \frac{x}{a^2}\widehat{yx}+\frac{y}{b^2}\widehat{yy}+\frac{z}{c^2}\widehat{yz} &= \frac{y}{b^2}(Sx^2+Ty^2+Uz^2) \\ \frac{x}{a^2}\widehat{zx}+\frac{y}{b^2}\widehat{zy}+\frac{z}{c^2}\widehat{zz} &= \frac{z}{c^2}(Sx^2+Ty^2+Uz^2) \end{aligned}\right\} \ldots\ldots(7).$$

The form of the surface equations is obvious when it is borne in mind that the direction cosines of the outward drawn normal are px/a^2, py/b^2 and pz/c^2, where p is the perpendicular from the centre on the tangent plane to the surface (1).

§ 4. The six equations (6) and (7) are insufficient to determine the fifteen constants appearing in (5), the reason being that these constants are not all independent.

The strains in an elastic solid, as is well known, have to satisfy certain "equations of compatibility," and this implies a corresponding series of equations for the stresses.

For the strains there are three equations of compatibility* of the type

$$\frac{d^2 s_z}{dy^2}+\frac{d^2 s_y}{dx^2}-\frac{d^2\sigma_{xy}}{dx\,dy}=0,$$

and three of the type

$$2\frac{d^2 s_x}{dy\,dz}+\frac{d^2\sigma_{yz}}{dx^2}-\frac{d^2\sigma_{zx}}{dx\,dy}-\frac{d^2\sigma_{xy}}{dx\,dz}=0.$$

By (4) these imply for the stresses three equations of compatibility of the type

$$\frac{d^2}{dy^2}\{\widehat{xx}-\eta(\widehat{yy}+\widehat{zz})\}+\frac{d^2}{dx^2}\{\widehat{yy}-\eta(\widehat{zz}+\widehat{xx})\} - 2(1+\eta)\frac{d^2\widehat{xy}}{dx\,dy}=0\ldots\ldots\ldots(8),$$

and three of the type

$$\frac{d^2}{dy\,dz}\{\widehat{xx}-\eta(\widehat{yy}+\widehat{zz})\} +(1+\eta)\left\{\frac{d^2\widehat{yz}}{dx^2}-\frac{d^2\widehat{zx}}{dx\,dy}-\frac{d^2\widehat{xy}}{dx\,dz}\right\}=0\ldots\ldots(9).$$

* See Todhunter and Pearson's *History of Elasticity*, Vol. II., Part I, p. 74; or Love's *Treatise on Elasticity*, Vol. I., p 122.

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§ 5. Substituting from (5) in the equations of the type (8), we obtain the three following relations between the constants of our solution

$$A_2' + B_2 - \eta(A_2 + C_2 + B_2' + C_2') - 2(1+\eta)N = 0 \quad \ldots\ldots(10),$$

$$A_2'' + C_2 - \eta(A_2 + B_2 + B_2'' + C_2'') - 2(1+\eta)M = 0 \quad \ldots\ldots(11),$$

$$B_2'' + C_2' - \eta(A_2' + B_2' + A_2'' + C_2'') - 2(1+\eta)L = 0 \quad \ldots\ldots(12).$$

The equations of the type (9) are identically satisfied.

From the equations (6) we have

$$A_2 = -(\tfrac{1}{2}P\rho + M + N) \quad \ldots\ldots\ldots\ldots(13),$$

$$B_2' = -(\tfrac{1}{2}Q\rho + N + L) \quad \ldots\ldots\ldots\ldots(14),$$

$$C_2'' = -(\tfrac{1}{2}R\rho + L + M) \quad \ldots\ldots\ldots\ldots(15).$$

These equations are to be regarded as expressing A_2, B_2', and C_2'' in terms of P, Q, R, L, M and N.

The first of the surface equations (7) will be found to contain a term in x, while the other terms are of the third degree in the variables x, y, z. As the equation holds over the surface of the ellipsoid, the simplest plan is to make it homogeneous by means of (1). It then takes the form

$$x\left[\frac{1}{a^2}\left\{A_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+A_2x^2+A_2'y^2+A_2''z^2\right\}\right.$$
$$\left.+2N\frac{y^2}{b^2}+2M\frac{z^2}{c^2}-\frac{1}{a^2}(Sx^2+Ty^2+Uz^2)\right]=0.$$

As this holds for all surface values of x, y, z, the coefficients of x^2, y^2, and z^2 inside the square bracket [ ] must separately vanish. In a similar way each of the two other surface equations supplies likewise three relations between the constants of the solution. We thus get the nine equations

$$a^{-4}A_0 + a^{-2}A_2 = a^{-2}S \quad \ldots\ldots\ldots\ldots(16),$$

$$a^{-2}b^{-2}A_0 + a^{-2}A_2' + 2b^{-2}N = a^{-2}T \quad \ldots\ldots(17),$$

$$a^{-2}c^{-2}A_0 + a^{-2}A_2'' + 2c^{-2}M = a^{-2}U \quad \ldots\ldots(18),$$

$$a^{-2}b^{-2}B_0 + b^{-2}B_2 + 2a^{-2}N = b^{-2}S \quad \ldots\ldots(19),$$

$$b^{-4}B_0 + b^{-2}B_2' = b^{-2}T \quad \ldots\ldots\ldots\ldots(20),$$

$$b^{-2}c^{-2}B_0 + b^{-2}B_2'' + 2c^{-2}L = b^{-2}U \quad \ldots\ldots(21),$$

$$c^{-2}a^{-2}C_0 + c^{-2}C_2 + 2a^{-2}M = c^{-2}S \quad \ldots\ldots(22),$$

$$b^{-2}c^{-2}C_0 + c^{-2}C_2' + 2b^{-2}L = c^{-2}T \quad \ldots\ldots(23),$$

$$c^{-4}C_0 + c^{-2}C_2'' = c^{-2}U \quad \ldots\ldots\ldots\ldots(24).$$

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§ 6. The most convenient procedure is to express all the other constants in terms of the three L, M, N, and the known quantities P, Q, R, S, T, U.

Thus, from (13) and (16), from (14) and (20), and from (15) and (24) respectively we find

$$\left.\begin{aligned} A_0 &= a^2(\tfrac{1}{2}P\rho + S + M + N) \\ B_0 &= b^2(\tfrac{1}{2}Q\rho + T + N + L) \\ C_0 &= c^2(\tfrac{1}{2}R\rho + U + L + M) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (25).$$

Then employing (25) in (17), (18), (19), (21), (22) and (23), we get

$$\left.\begin{aligned} A_2' &= T - \frac{a^2}{b^2}(\tfrac{1}{2}P\rho + S + M + 3N) \\ A_2'' &= U - \frac{a^2}{c^2}(\tfrac{1}{2}P\rho + S + 3M + N) \\ B_2 &= S - \frac{b^2}{a^2}(\tfrac{1}{2}Q\rho + T + 3N + L) \\ B_2'' &= U - \frac{b^2}{c^2}(\tfrac{1}{2}Q\rho + T + N + 3L) \\ C_2 &= S - \frac{c^2}{a^2}(\tfrac{1}{2}R\rho + U + L + 3M) \\ C_2' &= T - \frac{c^2}{b^2}(\tfrac{1}{2}R\rho + U + 3L + M) \end{aligned}\right\} \quad \ldots\ldots (26).$$

We have thus expressed all the constants A, B, C in terms of L, M, N, and the known quantities P, Q, R, S, T, U.

§ 7. Finally substituting for the A, B, C constants in (10), (11) and (12), we find to determine L, M, N the three simple equations

$$\begin{aligned} &(3b^4 + 2b^2c^2 + 3c^4)L + \{c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2)\}M \\ &\qquad + \{b^4 - \eta(a^2b^2 + b^2c^2 + 3c^2a^2)\}N \\ &= \eta a^2(b^2 + c^2)(\tfrac{1}{2}P\rho + S) + b^2[\tfrac{1}{2}Q\rho(\eta c^2 - b^2) + T\{c^2(1-\eta) - b^2\}] \\ &\qquad + c^2[\tfrac{1}{2}R\rho(\eta b^2 - c^2) + U\{b^2(1-\eta) - c^2\}] \quad \ldots\ldots (27), \end{aligned}$$

$$\begin{aligned} &\{c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2)\}L + (3c^4 + 2c^2a^2 + 3a^4)M \\ &\qquad + \{a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2)\}N \\ &= a^2[\tfrac{1}{2}P\rho(\eta c^2 - a^2) + S\{c^2(1-\eta) - a^2\}] + \eta b^2(c^2 + a^2)(\tfrac{1}{2}Q\rho + T) \\ &\qquad + c^2[\tfrac{1}{2}R\rho(\eta a^2 - c^2) + U\{a^2(1-\eta) - c^2\}] \quad \ldots\ldots (28), \end{aligned}$$

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$$\{b^4 - \eta(a^2b^2 + b^2c^2 + 3c^2a^2)\}L + \{a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2)\}M + (3a^4 + 2a^2b^2 + 3b^4)N$$
$$= a^2[\tfrac{1}{2}P\rho(\eta b^2 - a^2) + S\{b^2(1-\eta) - a^2\}] + b^2[\tfrac{1}{2}Q\rho(\eta a^2 - b^2) + T\{a^2(1-\eta) - b^2\}] + \eta c^2(a^2 + b^2)(\tfrac{1}{2}R\rho + U) \quad \ldots\ldots\ldots\ldots\ldots(29).$$

Let us put

$$\Pi = \begin{vmatrix} 3b^4 + 2b^2c^2 + 3c^4 & c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2) & b^4 - \eta(a^2b^2 + b^2c^2 + 3c^2a^2) \\ c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2) & 3c^4 + 2c^2a^2 + 3a^4 & a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2) \\ b^4 - \eta(a^2b^2 + b^2c^2 + 3c^2a^2) & a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2) & 3a^4 + 2a^2b^2 + 3b^4 \end{vmatrix} \quad \ldots(30).$$

This determinant is of the well known symmetrical form occurring in the discriminant of a quadric surface.

Denoting the minors by Π_{11}, Π_{12}, etc., we have, since

$$\Pi_{12} = \Pi_{21}, \text{ etc,}$$

$$L\Pi = a^2[\tfrac{1}{2}P\rho\{\eta(b^2 + c^2)\Pi_{11} + (\eta c^2 - a^2)\Pi_{12} + (\eta b^2 - a^2)\Pi_{13}\}$$
$$+ S\{\eta(b^2 + c^2)\Pi_{11} + (c^2(1-\eta) - a^2)\Pi_{12} + (b^2(1-\eta) - a^2)\Pi_{13}\}]$$
$$+ b^2[\tfrac{1}{2}Q\rho\{(\eta c^2 - b^2)\Pi_{11} + \eta(c^2 + a^2)\Pi_{12} + (\eta a^2 - b^2)\Pi_{13}\}$$
$$+ T\{(c^2(1-\eta) - b^2)\Pi_{11} + \eta(c^2 + a^2)\Pi_{12} + (a^2(1-\eta) - b^2)\Pi_{13}\}]$$
$$+ c^2[\tfrac{1}{2}R\rho\{(\eta b^2 - c^2)\Pi_{11} + (\eta a^2 - c^2)\Pi_{12} + \eta(a^2 + b^2)\Pi_{13}\}$$
$$+ U\{(b^2(1-\eta) - c^2)\Pi_{11} + (a^2(1-\eta) - c^2)\Pi_{12} + \eta(a^2 + b^2)\Pi_{13}\}]$$
$$\ldots\ldots\ldots\ldots(31).$$

The values of M and N are obtained from those of L by simply writing for the first suffix in the Π's 2 in the former case, and 3 in the latter, in place of 1. If $Sa^2 = Tb^2 = Uc^2$, then L, M and N are independent of the surface forces. This is obviously correct as the surface forces then reduce to a uniform traction Sa^2.

§ 8. The values of the Π's are easily calculated. Thus

$$\Pi_{11} = 8a^6 + 6a^4(b^2 + c^2) + 9(a^4b^4 + b^4c^4 + c^4a^4) + 4a^2b^2c^2 + 6a^2b^2c^2(b^2 + c^2) + 2\eta a^4(a^2b^2 + 3b^2c^2 + c^2a^2) - \eta^2(a^2b^2 + 3b^2c^2 + c^2a^2)^2 \ldots\ldots(32),$$

$$\Pi_{23} = b^4c^4 - a^4(3b^4 + 2b^2c^2 + 3c^4) + 2\eta b^2c^2\{2a^2(b^2 + c^2) + 4b^4 + 3b^2c^2 + 4c^4\} + \eta^2(a^2b^2 + b^2c^2 + 3c^2a^2)(3a^2b^2 + b^2c^2 + c^2a^2) \ldots(33);$$

and from these the other minors may be written down by symmetry.

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The determinant Π itself may be expressed in several convenient forms,* *e.g.*

$$\left.\begin{aligned}
\Pi &= (\Pi_{12}\Pi_{13} - \Pi_{11}\Pi_{23}) \div \{a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2)\} \\
&= (\Pi_{22}\Pi_{33} - \Pi_{23}^2) \div (3b^4 + 2b^2c^2 + 3c^4) \\
&= (3b^4 + 2b^2c^2 + 3c^4)(3c^4 + 2c^2a^2 + 3a^4)(3a^4 + 2a^2b^2 + 3b^4) \\
&\quad + 2\{c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2)\}\{b^4 - \eta(a^2b^2 + b^2c^2 \\
&\qquad\qquad + 3c^2a^2)\}\{a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2)\} \\
&\quad - (3a^4 + 2a^2b^2 + 3b^4)\{c^4 - \eta(3a^2b^2 + b^2c^2 + c^2a^2)\}^2 \\
&\quad - (3b^4 + 2b^2c^2 + 3c^4)\{a^4 - \eta(a^2b^2 + 3b^2c^2 + c^2a^2)\}^2 \\
&\quad - (3c^4 + 2c^2a^2 + 3a^4)\{b^4 - \eta(a^2b^2 + b^2c^2 + 3c^2a^2)\}^2
\end{aligned}\right\} \quad (34).$$

Substituting the values of Π and its minors in (31), we obtain the value of L explicitly, and similarly obtain the explicit values of M and N.

§ 9. The values of L, M, N being thus determined, we might proceed to determine the other twelve constants of the solution by means of the equations (13) to (15), (25) and (26), and thereafter substitute the values so determined in (5). It is simpler, however, to substitute in (5) in the first instance from equations (13) to (15), (25) and (26), thus obtaining general expressions for the stresses from which all the constants except L, M and N have been eliminated. These expressions are as follows:

$$\left.\begin{aligned}
\widehat{xx} &= Sx^2 + Ty^2 + Uz^2 + a^2\left[(\tfrac{1}{2}P\rho + S)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right. \\
&\qquad \left. + M\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + N\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)\right] \\
\widehat{yy} &= Sx^2 + Ty^2 + Uz^2 + b^2\left[(\tfrac{1}{2}Q\rho + T)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right. \\
&\qquad \left. + N\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) + L\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right)\right] \\
\widehat{zz} &= Sx^2 + Ty^2 + Uz^2 + c^2\left[(\tfrac{1}{2}R\rho + U)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right. \\
&\qquad \left. + L\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) + M\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right] \\
\widehat{yz} &= 2Lyz \\
\widehat{zx} &= 2Mzx \\
\widehat{xy} &= 2Nxy
\end{aligned}\right\} \quad (35).$$

* See Frost's *Solid Geometry*, 3rd edition, Art. 391.

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The existence of these forms renders any further reference unnecessary to the twelve A, B, C constants, and all that is required is to substitute in (35) the values of L, M, N from (31) and the two similar equations.

From the general expressions for the stresses we deduce those for the strains by means of (4). Thus

$$\left.\begin{aligned} s_x \equiv \frac{dx}{d_x} &= \frac{1}{E}\Big[(1-2\eta)(Sx^2+Ty^2+Uz^2) \\ &+ [a^2(\tfrac{1}{2}P\rho+S)-\eta\{b^2(\tfrac{1}{2}Q\rho+T)+c^2(\tfrac{1}{2}R\rho+U)\}] \\ &\qquad \times\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right) \\ &+ (a^2M-\eta b^2L)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{3z^2}{c^2}\right) \\ &+ (a^2N-\eta c^2L)\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}-\frac{z^2}{c^2}\right) \\ &- \eta(b^2N+c^2M)\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)\Big] \\ \sigma_{yz} \equiv \frac{d\beta}{dz}+\frac{d\gamma}{dy} &= \frac{4(1+\eta)}{E}Lyz \end{aligned}\right\} (36).$$

The values of the other strains can be written down from symmetry.

Simple integration suffices to determine the values of α from s_x, of β from s_y, and of γ from s_z. This cannot be taken for granted, because, for instance, α might have contained terms in y^2 and z^2 independent of x; and at first sight it might even appear incorrect, because the slides deduced from the displacements so found appear of a much more complicated form than σ_{yz} in (36). It is easily verified, however, that the numerous apparently redundant terms in the expression for any one of the slides vanish in virtue of one of the surface equations (27), 28) or (29).

It will suffice to record the expression for one of the displacements; thus

$$\alpha = \frac{x}{E}\Big[(1-2\eta)(\tfrac{1}{3}Sx^2+Ty^2+Uz^2)+[a^2(\tfrac{1}{2}P\rho+S) \\ -\eta\{b^2(\tfrac{1}{2}Q\rho+T)+c^2(\tfrac{1}{2}R\rho+U)\}]\left(1-\tfrac{1}{3}\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)$$

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$$+ (a^2M - \eta b^2L)\left(1 - \tfrac{1}{5}\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right)$$

$$+ (a^2N - \eta c^2L)\left(1 - \tfrac{1}{5}\frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)$$

$$- \eta\,(b^2N + c^2M)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\Big] \quad \ldots\ldots\ldots(37).$$

The expressions for β and γ can be written down from symmetry.

§ 10. The forms in which the expressions for the stresses, strains and displacements are presented may seem unusual. One of the principal reasons for grouping the terms in this way is that it enables the values at the surface to be very readily arrived at. It is thus very easily verified that the expressions for the stresses satisfy the surface conditions. We are also enabled to establish with extreme simplicity a very general relation between the effects of the bodily and surface forces in the case of incompressible material.

By *incompressible* material is meant material in which, while the rigidity and Young's modulus are finite, the bulk modulus is infinite. Thus, by (3) it is material for which

$$\eta = 0{\cdot}5.$$

For such material

$$\eta = 1 - \eta,$$

and so by (31) in L, M and N the coefficients of $\frac{1}{2}P\rho$, $\frac{1}{2}Q\rho$, and $\frac{1}{2}R\rho$ are the same as those of S, T and U respectively, Again, as the coefficient $1-2\eta$ in (36) and (37) vanishes. s_x and α are functions of $\frac{1}{2}P\rho + S$, $\frac{1}{2}Q\rho + T$, $\frac{1}{2}R\rho + U$ and of L, M, N. Also the shearing stresses and slides depend only on L, M and N. Thus, in an incompressible material the strains and displacements due to the bodily forces derived from the potential

$$V = \tfrac{1}{2}(Px^2 + Qy^2 + Rz^2)$$

are the same as those due to the purely normal surface *tractions*

$$\tfrac{1}{2}\rho\,(Px^2 + Qy^2 + Rz^2).$$

Another way of looking at the result is that the strains and displacements vanish when there act simultaneously the bodily forces derived from the potential V and the normal surface *pressures* ρV.

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The facts as regards the stresses are most easily seen from the latter point of view. Putting in (35)

$$S = -\tfrac{1}{2}P\rho,\ T = -\tfrac{1}{2}Q\rho,\ U = -\tfrac{1}{2}R\rho,$$

we get, as L, M and N vanish,

$$\widehat{xx} = \widehat{yy} = \widehat{zz} = -\tfrac{1}{2}\rho\,(Px^2 + Qy^2 + Rz^2)$$
$$= -\tfrac{1}{2}\rho V.$$

Thus the stresses, unlike the strains and displacements, do not vanish, but reduce at every point to three equal principal pressures $\frac{1}{2}\rho V$.

If V—in this case a negative quantity—represents the variable part of the potential for the mutual gravitation of the incompressible ellipsoid, we see that the strains and displacements arising from the gravitational action are identical with those arising from the normal surface *pressure* $(-\rho V)$. To obtain, however, the stress system in the gravitational problem, we must superpose on the stress system arising from the normal surface pressure $(-\rho V)$ a second stress system in which the three principal *tractions* at every point have the common value $(-\rho V)$.

As a second example of the application of the results to an incompressible material, suppose the equi-potential surfaces of the bodily forces—V = constant—to be similar and similarly situated to the surface of the ellipsoid. Then as V is constant over the surface, the strains and displacements due to the bodily forces are the same as those due to uniform normal surface force, and so vanish as the material is incompressible. The stress system of course does not vanish, but consists at every point of three equal principal tractions, the magnitude of which is constant over all surfaces concentric with, and similar and similarly situated to, the surface of the ellipsoid.

I shall only glance briefly at the general information the solution supplies relative to the action of normal surface forces in one or two special forms of surface.

Very oblate ellipsoid.

§ 11. Suppose $2c$ the short axis, and retain only the lowest powers of c/a and c/b occurring in each expression or coefficient. Then we get

$$\left.\begin{aligned}
\Pi &= 6\,(1-\eta^2)\,a^4b^4\,(4a^4 + 3a^2b^2 + 4b^4 + \eta a^2b^2)\\
\Pi_{11} &= a^4\,(8a^4 + 6a^2b^2 + 9b^4 + 2\eta a^2b^2 - \eta^2 b^4)\\
\Pi_{22} &= b^4\,(9a^4 + 6a^2b^2 + 8b^4 + 2\eta a^2b^2 - \eta^2 a^4)\\
\Pi_{12} &= a^2b^2\,[a^2b^2\,(1+\eta^2) + 2\eta\,(4a^4 + 3a^2b^2 + 4b^4)]\\
\Pi_{23}/3 &= -\Pi_{13} = -\Pi_{22} = 3\,(1-\eta^2)\,a^4b^4
\end{aligned}\right\} \ldots(38).$$

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Substituting in (31) and the corresponding expression for N, we find

$$L\Pi = (1-\eta^2)\, a^4b^2 \left[Sa^2b^2\,(2a^2-3b^2+4\eta b^2) - Tb^2\,(8a^4+9a^2b^2+6b^4-2\eta a^2b^2) + Uc^2\,\{8a^4+7a^2b^2+9b^4-2\eta b^2\,(a^2+2b^2)\}\right] \ldots\ldots(39),$$

$$N\Pi = 3\,(1-\eta^2)\, a^4b^4 \left[Sa^2\,(3b^2-2a^2-4\eta b^2) + Tb^2\,(3a^2-2b^2-4\eta a^2) - Uc^2\,(1-4\eta)\,(a^2+b^2)\right] \ldots(40).$$

The value of M is got from that of L by interchanging S with T and a with b.

Substituting their values for L, M, N in (35), (36) and (37), we get the corresponding values of the stresses, strains and displacements. The expressions do not simplify very much and are rather long, so it seems unnecessary to actually write them down. The conclusions we are about to deduce for the stresses will be easily followed when regard is paid to the form of the results for L and N.

It appears that in $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, and $\widehat{xy}$ alike,* assuming b and a of the same order of magnitude,

the principal terms in S are of the order Sa^2,
" " T " Tb^2,
" " U " Uc^2,

whereas the corresponding terms in $\widehat{yz}$ and $\widehat{zx}$ bear to these ratios of the order $c:a$ or $c:b$.

Thus, in the case whether of S, T or U forces, $\widehat{yz}$ and $\widehat{zx}$ may for a first approximation be neglected, but $\widehat{zz}$ is of the same order of magnitude as $\widehat{xx}$, $\widehat{yy}$ or $\widehat{xy}$.

Differentiation with respect to a variable reduces by unity the index of the power of that variable in any term containing it. Thus, in the case of the product zx, the differential coefficient with respect to x is negligible compared to that with respect to z; and consequently $\frac{d\widehat{zx}}{dx}$ is negligible compared to $\frac{d\widehat{zx}}{dz}$.

* A term such as Sxy is regarded as of the order Sab or Sa^2, this being the order of its mean value; but it vanishes of course for special points.

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The stresses $\widehat{xx}$ and $\widehat{xy}$, however, are large compared to $\widehat{zx}$, and their differential coefficients $\frac{d\widehat{xx}}{dx}$ and $\frac{d\widehat{xy}}{dy}$ are not negligible compared to $\frac{d\widehat{zx}}{dz}$, but are precisely of the same order of magnitude.

Similarly $\frac{d\widehat{yy}}{dy}$ and $\frac{d\widehat{xy}}{dx}$ are of the same order of magnitude as $\frac{d\widehat{yz}}{dz}$.

Even in differentiating a single stress it must be borne in mind that differentiation with respect to z may operate on a different term from differentiation with respect to x, say. If the terms operated on are originally of the same order of magnitude, then the differential coefficient of the stress with respect to x is relatively negligible. Thus, when the surface forces Uz^3 alone act, the terms in x^3 and z^3 in $\widehat{xx}$ are of the same order of magnitude, and $\frac{d\widehat{xx}}{dx}$ is negligible compared to $\frac{d\widehat{xx}}{dz}$.

If, however, the terms operated on be of different orders of magnitude, no such conclusion necessarily follows. An example of this latter case arises when the surface forces are confined to $Sx^3 + Ty^3$. The principal terms containing x^3 in $\widehat{xx}$ are of the order Sx^3 or Tx^3, but the coefficients of the principal terms in z^1—which would be of the order Sa^3z^2/c^2 or Tb^3z^2/c^2—vanish, and the most important terms left are at most of the order Sz^3 or Tz^3. Hence, in this case it is $\frac{d\widehat{xx}}{dz}$ that is negligible relatively to $\frac{d\widehat{xx}}{dx}$.

Similar results apply of course for differentiation with respect to y.

It is also possible for a stress to be a function of x and y, and yet absolutely independent of z, as, for example, $\widehat{xy}$ in the present problem. In such a case the differential coefficient with respect to z is absolutely nil.

A flat ellipsoid, however small c/a and c/b may be, is not a plane plate, and the system of forces we are considering is admittedly of a very special kind. Still, the results obtained above will, I think, repay the study of those

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interested in the theory of thin plates, as suggestive at least of possibilities.*

Before quitting the subject of the flat ellipsoid I would only call attention to the fact that for a first approximation $\widehat{zz}$ is given by the simple formula

$$\widehat{zz} = Sx^2 + Ty^2 + Uc^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Very elongated ellipsoid.

§ 12. By this is meant a form in which a/c and b/c are very small. Its central portion approximates closely to a cylinder whose section is an ellipse of semi-axes a and b.

As the "principal" terms in certain of the coefficients in the expressions for the stresses cut out, it is necessary to make use in some cases of "secondary" terms. I thus give the terms in the second highest powers of c in the following expressions for the Π's:

$$\left.\begin{aligned}
\Pi &= 8c^8(1-\eta^2)(3a^4+2a^2b^2+3b^4) + 2c^6(1-\eta^2)(a^2+b^2) \\
&\qquad \times \{(3+\eta)(3a^4+2a^2b^2+3b^4) + 8\eta a^2b^2\} \\
\Pi_{11} &= c^4\{3(3a^4+2a^2b^2+3b^4) - \eta^2(a^2+3b^2)^2\} \\
&\quad + 2c^2a^2\{3a^4+2a^2b^2+3b^4+\eta a^2(a^2+3b^2) - \eta^2b^2(a^2+3b^2)\} \\
\Pi_{22} &= 8c^8 + 2c^6(3+\eta)(a^2+b^2) \\
\Pi_{12} &= -c^4\{3a^4+2a^2b^2+3b^4 - \eta^2(3a^2+b^2)(a^2+3b^2)\} \\
&\qquad + 4c^2\eta(1+\eta)a^2b^2(a^2+b^2) \\
\Pi_{13} &= 8\eta c^6a^2 + c^4\{a^4 - 3b^4 + 2\eta a^2(3a^2+2b^2) \\
&\qquad + \eta^2(a^2+b^2)(a^2+3b^2)\}
\end{aligned}\right\} \quad (41).$$

Π_{22} and Π_{23} may be derived from Π_{11} and Π_{13} respectively by interchanging a and b.

Substituting these values in the expressions for L and N, we find, after reduction, as first approximations,

$$L = -S\frac{a^2}{c^2}\left[\tfrac{1}{8} - \frac{\eta(a^2+b^2)(a^2+3b^2)}{2(3a^4+2a^2b^2+3b^4)}\right] + T\frac{b^2}{c^2}\left[\tfrac{3}{8} - \frac{\eta(a^2+b^2)(a^2+3b^2)}{2(3a^4+2a^2b^2+3b^4)}\right] - \tfrac{1}{4}U \quad \ldots\ldots(42),$$

$$N = \frac{(a^2-b^2)(Tb^2-Sa^2) + \tfrac{1}{2}U\{a^4+b^4+\frac{2\eta}{1+\eta}a^2b^2\}}{3a^4+2a^2b^2+3b^4} \quad \ldots\ldots(43).$$

* See Todhunter and Pearson's *History of Elasticity*, Vol. II., Part I., Art. 386 *et seq.*, also Part II., Art. 1140.

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The value of M is got from that of L by interchanging a with b and S with T.

The simplicity of the first approximation values of L and M when η is zero is noteworthy.

If the maximum values of the S, T and U forces be of the same order of magnitude, then Sa^2, Tb^2, Uc^2 are of the same order. In this case the principal terms involving S, T and U in L and M are of the same order of magnitude, while, excluding the spheroid $a = b$, the principal terms involving U in N are negligible compared to those involving S and T.

When their values are substituted for L, M and N in the expressions for the stresses it will be found, according as we consider the S, T or U forces, that

$\widehat{xx}$ and $\widehat{yy}$ are of the order Sa^2, Tb^2 or Uz^2,

$\widehat{zz}$ is „ „ Sa^2, Tb^2 or Uc^2,

$\widehat{xy}$ „ „ Sxy, Txy or Uxy,

$\widehat{yz}$ „ „ $S\frac{a^2}{c^2}yz$, $T\frac{b^2}{c^2}yz$ or Uyz,

$\widehat{zx}$ „ „ $S\frac{a^2}{c^2}zx$, $T\frac{b^2}{c^2}zx$ or Uzx.

As regards the S and T forces $\widehat{xx}$, $\widehat{yy}$ and $\widehat{xy}$ are not small compared to $\widehat{zz}$, but of the same order of magnitude, and they are large compared to $\widehat{yz}$ and $\widehat{zx}$.

As regards the U forces when z is of the same order as c, $\widehat{xy}$ is small compared to $\widehat{yz}$ and $\widehat{zx}$, but these in turn are small compared to $\widehat{xx}$ and $\widehat{yy}$, whose values are of the same order of magnitude as that of $\widehat{zz}$. Near the centre of the ellipsoid, however, z/c is small, and $\widehat{zz}$ is large compared to all the other stresses.

For the differential coefficients of the stresses it will be found, according as we consider S, T or U, that

$\frac{d\widehat{xx}}{dx}$ and $\frac{d\widehat{xy}}{dy}$ are of the order Sx, Tx or Ux,

while $\frac{d\widehat{zx}}{dz}$ is „ $\frac{a^2}{c^2}Sx$, $\frac{b^2}{c^2}Tx$ or Ux.

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Thus in the case of the S and T forces $\widehat{\frac{dzx}{dz}}$ is negligible compared to $\widehat{\frac{dxx}{dx}}$ and $\widehat{\frac{dxy}{dy}}$, but in the case of the U forces it is not so.

Analogous results hold for $\widehat{\frac{dxy}{dx}}$, $\widehat{\frac{dyy}{dy}}$ and $\widehat{\frac{dyz}{dz}}$.

In the case whether of S, T or U forces $\widehat{\frac{dzx}{dx}}$, $\widehat{\frac{dyz}{dy}}$ and $\widehat{\frac{dzz}{dz}}$ are of the same order of magnitude.

The resemblance of a very elongated ellipsoid to a long bar of uniform section is fairly close, and the above results certainly suggest that at least in cases where the load on a long bar is not confined to the ends the neglect of one set of stresses compared to another, or of the differential coefficients of any of the stresses in the body-stress equations, is a risky process.* The ordinary fundamental assumption, it will be remembered, in the theory of thin bars is that $\widehat{xx}$, $\widehat{yy}$ and $\widehat{xy}$ are negligible compared to the other stresses, whereas in the case of the present problem the whole three, as we have seen, may be large compared to $\widehat{yz}$ or $\widehat{zx}$, and not small compared to $\widehat{zz}$.

NOTE ON MR. CULLOVIN'S DEMONSTRATION OF THE THEORY OF PARALLELS.

By A. E. H. Love.

IN a short paper in the *Quarterly Journal*, No. 106 (August, 1894), Mr. Cullovin has attempted to give a rigorously Euclidean demonstration of Euclid's parallel axiom. The remaining axioms and the postulates are assumed, and all the propositions in the First Book up to and including the 26th, and it is proposed to deduce the parallel axiom without assuming anything else, and without an appeal to intuition. This is what is meant by a demonstration of the axiom, and it is well known to mathematicians that all attempts to give such a demonstration are foredoomed to failure, inasmuch as in the abstract logical system in which Geometry becomes a body of reasoning deduced from definitions and hypotheses

* See Todhunter and Pearson's *History*, Vol. II. Part. II., pp, 189 *et seq.*

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suggested by intuition, the place here indicated is precisely a place at which a choice can be made among various logical possibilities, and Euclid's axiom states the choice that is pointed out by ordinary intuition; *i.e.* this axiom is apparently guaranteed by experience, but is not a logical necessity. Any such statement is properly an axiom. A demonstration of the axiom in the sense described above would show that, as an axiom, the statement is unnecessary. It is also well known to mathematicians that Euclid's statement is not the only one from which his theory of parallels can be obtained, but that there are numerous equivalent forms of statement involving practically the same kind of appeal to intuition as his axiom does. It is the purpose of this note to show that Mr. Cullovin's demonstration is unsatisfactory without an appeal to intuition, or, in other words, that an axiom which is not in Euclid's system is required to complete his demonstration.

In Mr. Cullovin's figure (fig. 43) YZ is a line at right angles to BD, and BC is a line making an acute angle with BD. It is to be shown that BC and YZ meet. Taking L, any point on BC, and drawing LK at right angles to BD, the triangle BKL is half of an isosceles triangle, the other symmetrical half being BKF. Thus, if points K, I, ... are taken on BD, and lines drawn from them at right angles to BD, some of these lines meet BC, and it is clear, as proved by Mr. Cullovin, that, if the perpendicular to BD at I meets BC, then all the perpendiculars to BD, which go from points of it between B and I, also meet BC. If the perpendicular YZ does not meet BC, then there must be a point Q between B and Y such that all the perpendiculars which go from points between B and Q meet BC, and no perpendicular which goes from a point on the side of Q remote from B meets BC. Mr. Cullovin assumes that it follows from this that the perpendicular from Q is the last which meets BC, and then proves easily that this is impossible.

But there is no reason why there should be a last perpendicular which meets BC. It is just as possible, logically, that the perpendicular from Q should be the first which fails to meet BC. If this were the case, then the series of perpendiculars that go out from points K, I, ... between B and Q would meet BC in points L, H, ..., which lie further and further from B along BC, in such a way that, as I approaches Q, the length BH increases without limit, and yet, however great BH is taken, the perpendicular from H on BD always falls between B and Q. Since there is no last point of the segment BQ before Q is reached, there is no last perpendicular meeting

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BC. The assumption of a last perpendicular which meets BC is essentially different from the assumption of a first perpendicular which fails to meet BC.

Mr. Cullovin has attempted to answer this objection to his demonstration in a note inserted in the *Quarterly Journal*, No. 107 (March, 1895). He says "The lines which meet cannot intermingle with those supposed to fail, and no portion or point of the bisector can remain neutral. Consequently, the last line meeting would coincide with the first line failing, as no portion of the bisector could possibly intervene. In other words there can be no first line failing."

These words assume that the series of perpendiculars meeting BC cannot terminate without there being a "last line meeting." This assumption is not grounded in logic. We have in the line BD a continuous series of points having a certain property, viz., that the perpendiculars to BD through them meet BC, and this series can be assumed to be terminated by a point Q which has not that property. As already remarked, there is in the series of points no last point except Q, *i.e.*, the last point is one which is assumed not to have the property in question, not one which has it. The series of points terminates accordingly in a first which fails, not in a last which succeeds. So the series of lines may be assumed to terminate in a "first line failing," not in a "last line meeting."

Mr. Cullovin must be judged to have failed to prove that the assumptions are identical, and to have failed to exclude the assumption of a "first line failing." To do this would require an axiom equivalent to Euclid's.

Some further light may be thrown on the objection here raised to Mr. Cullovin's demonstration by considering other cases of continuous series of objects or forms, some of which have a certain property while others have not. It will appear that the transition from forms which have the property to forms which have not may involve a first which fails or a last which succeeds.

(i) A pencil of rays (fig. 44) being drawn from a point O, some of the rays meet a given line AB, while others do not. In the continuous series of rays of which OA, OB, OC, OX, OK are members, the ray OX, which is parallel to AB, is the first that fails to meet AB, and there is no last ray that meets AB.

(ii) The series $1^{-x}+2^{-x}+3^{-x}+\ldots$ is convergent for all values of $x>1$, but when $x=1$ it becomes divergent. In the continuous series of diminishing values of x, beginning say

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from $x = 2$, there is a first value for which the series fails to converge, but no last value for which it converges.

(iii) The series $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$ is convergent for $x < 1$ and for $x = 1$, but divergent for $x > 1$. In the continuous series of increasing values of x, beginning say from $x = 0$, there is a last value for which the series converges, but no first value for which it fails to converge.

ON CERTAIN GENERAL PROPERTIES OF POINT TRANSFORMATIONS.

By J. Brill, M.A., St. John's College.

1. SUPPOSE a point transformation in a plane defined by the equations

$$\phi(X, Y, x, y) = 0, \ \psi(X, Y, x, y) = 0.$$

Let P denote the point (x, y), and P' the corresponding point (X, Y). Then if Q be a neighbouring point to P, and Q' its corresponding point, we may consider the elemental straight line PQ as defining a direction through P, and $P'Q'$ as defining the corresponding direction through P'. We will take m to denote the tangent of the angle made by PQ with the axis of x, and M to denote the tangent of that made by $P'Q'$ with the axis of X. Then we have

$$M = \frac{dY}{dX} = \frac{\dfrac{\partial Y}{\partial x} + m\dfrac{\partial Y}{\partial y}}{\dfrac{\partial X}{\partial x} + m\dfrac{\partial X}{\partial y}}.$$

Now so long as the point P remains fixed, the quantities

$$\frac{\partial X}{\partial x}, \ \frac{\partial X}{\partial y}, \ \frac{\partial Y}{\partial x}, \ \frac{\partial Y}{\partial y}$$

remain unchanged in value, and the relation between M and m is of the homographic type. Hence if m_1, m_2, m_3, m_4 refer to any four directions through the point P, and M_1, M_2, M_3, M_4 to the corresponding directions through the point P', we have

$$\frac{(M_1 - M_2)(M_3 - M_4)}{(M_1 - M_3)(M_2 - M_4)} = \frac{(m_1 - m_2)(m_3 - m_4)}{(m_1 - m_3)(m_2 - m_4)}.$$

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Hence we see that the anharmonic ratio of the pencil formed by the four transformed directions is equal to that of the pencil formed by the four original ones.

Transformations will in general involve singular points where this property breaks down, as for instance in the well-known special case of transformations with the aid of conjugate functions.

2. We now proceed to discuss the similar property of a three-dimensional point transformation. Suppose the transformation to be defined by the equations

$$\phi(X, Y, Z, x, y, z) = 0,$$
$$\psi(X, Y, Z, x, y, z) = 0,$$
$$\chi(X, Y, Z, x, y, z) = 0;$$

and let P denote the point (x, y, z), and P' the transformed point (X, Y, Z). Further let Q be a neighbouring point to P, and Q' its transformed point, then PQ and $P'Q'$ define two corresponding straight lines. We will take four other neighbouring points to P, viz. R_1, R_2, R_3, R_4, and their corresponding points R_1', R_2', R_3', R_4', and will consider the four planes $P'Q'R_1'$, $P'Q'R_2'$, $P'Q'R_3'$, $P'Q'R_4'$ as corresponding to PQR_1, PQR_2, PQR_3, PQR_4, respectively.

Now let dx, dy, dz denote the projections on the axes of PQ, dx_1, dy_1, dz_1, those of PR_1, dx_2, dy_2, dz_2 those of PR_2, and so on. Then taking ξ, η, ζ for current coordinates, the equation of the plane PQR_1 is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx\ , & dy\ , & dz \\ dx_1\ , & dy_1\ , & dz_1 \end{vmatrix} = 0,$$

and that of PQR_4 is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx\ , & dy\ , & dz \\ dx_4\ , & dy_4\ , & dz_4 \end{vmatrix} = 0.$$

Moreover the equation of PQR_2 can be written in the form

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx\ , & dy\ , & dz \\ dx_1\ , & dy_1\ , & dz_1 \end{vmatrix} - k_1 \begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx\ , & dy\ , & dz \\ dx_4\ , & dy_4\ , & dz_4 \end{vmatrix} = 0,$$

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and that of PQR_3 in a similar form, k_3 being substituted for k_1 in the above expression.

Further k_1/k_3 will be the value of one of the anharmonic ratios of the four planes. Also, since PQR_3 passes through R_2, we obtain

$$k_1 = \frac{\begin{vmatrix} dx_2, & dy_2, & dz_2 \\ dx, & dy, & dz \\ dx_1, & dy_1, & dz_1 \end{vmatrix}}{\begin{vmatrix} dx_2, & dy_2, & dz_2 \\ dx, & dy, & dz \\ dx_4, & dy_4, & dz_4 \end{vmatrix}}.$$

Now

$$\begin{vmatrix} dX_2, & dY_2, & dZ_2 \\ dX, & dY, & dZ \\ dX_1, & dY_1, & dZ_1 \end{vmatrix} = J \begin{vmatrix} dx_2, & dy_2, & dz_2 \\ dx, & dy, & dz \\ dx_1, & dy_1, & dz_1 \end{vmatrix},$$

where

$$J = \frac{\partial (X, Y, Z)}{\partial (x, y, z)},$$

and similarly for other expressions of similar form. Thus we have

$$\frac{\begin{vmatrix} dX_2, & dY_2, & dZ_2 \\ dX, & dY, & dZ \\ dX_1, & dY_1, & dZ_1 \end{vmatrix}}{\begin{vmatrix} dX_2, & dY_2, & dZ_2 \\ dX, & dY, & dZ \\ dX_4, & dY_4, & dZ_4 \end{vmatrix}} = \frac{\begin{vmatrix} dx_2, & dy_2, & dz_2 \\ dx, & dy, & dz \\ dx_1, & dy_1, & dz_1 \end{vmatrix}}{\begin{vmatrix} dx_2, & dy_2, & dz_2 \\ dx, & dy, & dz \\ dx_4, & dy_4, & dz_4 \end{vmatrix}}.$$

Now, if K_1 and K_3 perform similar offices for the planes $P'Q'R_2'$ and $P'Q'R_3'$ that k_1 and k_3 perform for PQR_2 and PQR_3, then the above equation may be written

$$K_1 = k_1.$$

Similarly, we obtain

$$K_3 = k_3,$$

and therefore we have

$$\frac{K_1}{K_3} = \frac{k_1}{k_3},$$

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or the anharmonic ratio of the four planes $P'Q'R_1'$, $P'Q'R_2'$, $P'Q'R_3'$, $P'Q'R_4'$ is equal to that of the four planes PQR_1, PQR_2, PQR_3, PQR_4.

3. A similar property also holds in the case of an n-dimensional point transformation. In this case we will take $x_1, x_2, \ldots, x_n$ for the coordinates of the point P, and $X_1, X_2, \ldots, X_n$ for those of the transformed point P'. We will also fix $n-2$ neighbouring points to P, and denote them by the symbols $Q_1, Q_2, \ldots, Q_{n-2}$.

We will suppose the projections on the axes of PQ_1, $PQ_2, \ldots, PQ_{n-2}$ to be denoted respectively by

$$\begin{array}{l} dx_{1,1}\ ,\ dx_{2,1}, dx_{3,1}, \ldots, dx_{n,1}\ , \\ dx_{1,2}\ ,\ dx_{2,2}, dx_{3,2}, \ldots, dx_{n,2}\ , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ dx_{1,n-2}, \ldots\ldots\ldots\ldots\ldots, dx_{n,n-2}. \end{array}$$

We will also take four other neighbouring points, viz., R_1, R_2, R_3, R_4, and denote the projections of PR_1, PR_2, PR_3, PR_4 on the axes by the symbols

$$\begin{array}{l} dy_1, dy_2, \ldots, dy_n, \\ dz_1, dz_2, \ldots, dz_n, \\ ds_1, ds_2, \ldots, ds_n, \\ dt_1, dt_2, \ldots, dt_n. \end{array}$$

We will, as before, use dashed symbols to denote the transformed points.

Now $PQ_1Q_2 \ldots Q_{n-2}R_1$, $PQ_1Q_2 \ldots Q_{n-2}R_2$, $PQ_1Q_2 \ldots Q_{n-2}R_3$, and $PQ_1Q_2 \ldots Q_{n-2}R_4$, will define four loci of the first order, or n-point loci according to Buchheim's nomenclature.* We will consider the four n-point loci $P'Q_1'Q_2' \ldots Q_{n-2}'R_1'$, &c. as corresponding to the former four.

Taking $\xi_1, \xi_2 \ldots, \xi_n$ as current coordinates, the equation of the locus $PQ_1Q_2 \ldots Q_{n-2}R_1$ can be expressed in the form

$$\left| \begin{array}{l} \xi_1 - x_1, \ \xi_2 - x_2, \ \ldots, \ \xi_n - x_n \\ dx_{1,1}\ ,\ dx_{2,1}\ ,\ \ldots, dx_{n,1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dx_{1,n-2}, \ dx_{2,n-2}, \ \ldots, dx_{n,n-2} \\ dy_1\ ,\ dy_2\ ,\ \ldots, dy_n \end{array} \right| = 0,$$

which for brevity we will write $U = 0$.

* *Proc. London Mathematical Society*, Vol. XVI., p. 69.

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Similarly we can write down the equation of the locus $PQ_1Q_2 \dots Q_{n-2}R_4$, which we will write $V=0$.

Now the equations of the loci $PQ_1Q_2 \dots Q_{n-2}R_2$, $PQ_1Q_2 \dots Q_{n-2}R_3$, can be written in the forms

$$U - k_1 V = 0, \quad U - k_2 V = 0.$$

Hence, since the locus $PQ_1Q_2 \dots Q_{n-2}R_2$ contains the point R_2, we obtain

$$k_1 = \frac{\begin{vmatrix} dz_1 & , dz_2 & , \dots, dz_n \\ dx_{1,1} & , dx_{2,1} & , \dots, dx_{n,1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ dx_{1,n-2} & , dx_{2,n-2} & , \dots, dx_{n,n-2} \\ dy_1 & , dy_2 & , \dots, dy_n \end{vmatrix}}{\begin{vmatrix} dz_1 & , dz_2 & , \dots, dz_n \\ dx_{1,1} & , dx_{2,1} & , \dots, dx_{n,1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ dx_{1,n-2} & , dx_{2,n-2} & , \dots, dx_{n,n-2} \\ dt_1 & , dt_2 & , \dots, dt_n \end{vmatrix}}.$$

Now we have

$$\begin{vmatrix} dZ_1 & , dZ_2 & , \dots, & dZ_n \\ dX_{1,1} & , dX_{2,1} & , \dots, & dX_{n,1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ dX_{1,n-2} & , \dots\dots & , & dX_{n,n-2} \\ dY_1 & , dY_2 & , \dots, & dY_n \end{vmatrix} = J \begin{vmatrix} dz_1 & , dz_2 & , \dots, & dz_n \\ dx_{1,1} & , dx_{2,1} & , \dots, & dx_{n,1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ dx_{1,n-2} & , \dots\dots & , & dx_{n,n-2} \\ dy_1 & , dy_2 & , \dots, & dy_n \end{vmatrix},$$

where $$J = \frac{\partial(X_1, X_2, \dots, X_n)}{\partial(x_1, x_2, \dots, x_n)},$$

and similarly for the other expressions of the same type. Thus, if K_1 and K_2 perform the same offices for $P'Q_1'Q_2' \dots Q'_{n-2}R_2'$ and $P'Q_1'Q_2' \dots Q'_{n-2}R_3'$ that k_1 and k_2 do for the loci $PQ_1Q_2 \dots Q_{n-2}R_2$ and $PQ_1Q_2 \dots Q_{n-2}R_3$, then we have

$$K_1 = k_1 \text{ and } K_2 = k_2,$$

and therefore

$$\frac{K_1}{K_2} = \frac{k_1}{k_2}.$$

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Consider the four n-point loci

$$A_1(\xi_1 - x_1) \quad +\ldots\ldots\ldots+ A_n(\xi_n - x_n) \quad = 0 \ldots(1),$$

$$(A_1 - kB_1)(\xi_1 - x_1) + \ldots\ldots\ldots + (A_n - kB_n)(\xi_n - x_n) = 0 \ldots(2),$$

$$(A_1 - k'B_1)(\xi_1 - x_1) + \ldots\ldots\ldots + (A_n - k'B_n)(\xi_n - x_n) = 0 \ldots(3),$$

$$B_1(\xi_1 - x_1) \quad +\ldots\ldots\ldots+ B_n(\xi_n - x_n) \quad = 0 \ldots(4).$$

Each of the loci (2) and (3) will contain all the points common to the loci (1) and (4).

Further, consider the two-point locus

$$\frac{\xi_1 - \alpha_1}{l_1} = \frac{\xi_2 - \alpha_2}{l_2} = \ldots\ldots = \frac{\xi_n - \alpha_n}{l_n} = v \quad \ldots\ldots\ldots(5),$$

where $$l_1^2 + l_2^2 + \ldots\ldots + l_n^2 = 1.$$

Take E, F, G, H to denote the points in which (1), (2), (3), (4) are respectively met by (5). Also let v_1, v_2, v_3, v_4 denote the values of v corresponding to the points E, F, G, H. Then taking EF to denote the n-dimensional distance between the points E and F, we have

$$\frac{EF.GH}{EG.FH} = \frac{(v_3 - v_1)(v_4 - v_2)}{(v_3 - v_1)(v_4 - v_2)}.$$

If we write

$$x_1 - \alpha_1 = \lambda_1,\ x_2 - \alpha_2 = \lambda_2,\ \ldots,\ x_n - \alpha_n = \lambda_n,$$

we obtain

$$v_1 = \frac{\Sigma A\lambda}{\Sigma Al}, \quad v_2 = \frac{\Sigma (A - kB)\lambda}{\Sigma (A - kB) l},$$

$$v_3 = \frac{\Sigma (A - k'B)\lambda}{\Sigma (A - k'B) l}, \quad v_4 = \frac{\Sigma B\lambda}{\Sigma Bl}.$$

From these values we easily deduce

$$v_2 - v_1 = k \frac{\Sigma (A_p B_q - A_q B_p)(\lambda_p l_q - \lambda_q l_p)}{\Sigma Al . \Sigma (A - kB) l},$$

$$v_4 - v_3 = -\frac{\Sigma (A_p B_q - A_q B_p)(\lambda_p l_q - \lambda_q l_p)}{\Sigma Bl . \Sigma (A - k'B) l},$$

$$v_3 - v_1 = k' \frac{\Sigma (A_p B_q - A_q B_p)(\lambda_p l_q - \lambda_q l_p)}{\Sigma Al . \Sigma (A - k'B) l},$$

$$v_4 - v_2 = -\frac{\Sigma (A_p B_q - A_q B_p)(\lambda_p l_q - \lambda_q l_p)}{\Sigma Bl . \Sigma (A - kB) l}.$$

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Hence, it follows that

$$\frac{EF.GH}{EG.FH} = \frac{k}{k'},$$

and is therefore independent of the constants involved in (5). Taking this as the definition of an extension of anharmonic ratio, we see that the anharmonic ratio of the four transformed n-point loci, considered above, is equal to that of the four original loci.

ON A THEOREM OF JACOBI IN DYNAMICS.

By Prof. A. C. Dixon.

IN Dr. Routh's article under the above heading (*Quarterly Journal*, Vol. XXIII., p. 34) it is proved that "the motion of the axis of a top can be represented by the motion of the conjugate line of a body moving about a fixed point under no forces." As Jacobi's theorem relates to the motion of the top, and not only to that of its axis, it seems desirable to examine whether Dr. Routh's method will give this also. We shall use his notation.

We see from Dr. Routh's article that the curve of intersection of the two surfaces

$$a_1x^2 + b_1y^2 + c_1z^2 = 1,$$

$$a_2x^2 + b_2y^2 + c_2z^2 = 1,$$

can be a polhode for two Poinsot motions, which will be real if certain inequalities are satisfied, and which are to be found by solving the equations

$$\frac{(a_1 + \lambda a_2)^2}{a_1 + \mu a_2} = \frac{(b_1 + \lambda b_2)^2}{b_1 + \mu b_2} = \frac{(c_1 + \lambda c_2)^2}{c_1 + \mu c_2}$$

for λ and μ.

If A, B, C are the moments of inertia of a body capable of the one Poinsot motion under no forces, and A', B', C' those of a body capable of the other, then

$$\frac{A}{B-C} + \frac{A'}{B'-C'} = 0 = \frac{B}{C-A} + \frac{B'}{C'-A'} = \frac{C}{A-B} + \frac{C'}{A'-B'}.$$

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If, again, $k\omega_1$, $k\omega_2$, $k\omega_3$, kv are the angular velocities about the principal axes of the body, and the velocity along the polhode of the end of the instantaneous axis, in the first Poinsot motion, then in the second they will be at the same point of the polhode $k'\omega_1$, $k'\omega_2$, $k'\omega_3$, $-k'v$, k, k' being quite arbitrary constants, dependent upon the magnitudes of the initial impulses. We take $k=1$, $k'=-1$, so that the two instantaneous axes travel together along the polhode, and suppose the two bodies, rigidly connected together, to execute their respective Poinsot motions, the first relatively to a fixed space, the second with respect to a space S.

The motion of the space S is then to be compared with that of a top.

Now the angular velocities of S, relatively to the moving principal axes common to the two bodies, are ω_1, ω_2, ω_3 about those axes respectively, and those of the axes about themselves are ω_1, ω_2, ω_3 also.

Hence the space S has angular velocities $2\omega_1$, $2\omega_2$, $2\omega_3$ about axes which are themselves moving with angular velocities ω_1, ω_2, ω_3.

Let (l, m, n) be the direction cosines of any axis moving with S, and let

$$\Omega = 2(l\omega_1 + m\omega_2 + n\omega_3).$$

Then Ω is the angular velocity of S about this axis, and

$$\dot{\Omega} = 2(l\dot{\omega}_1 + m\dot{\omega}_2 + n\dot{\omega}_3) + 2(\dot{l}\omega_1 + \dot{m}\omega_2 + \dot{n}\omega_3).$$

But, since the axis (l, m, n) moves with S,

$$\dot{l} = n\omega_2 - m\omega_3,$$
$$\dot{m} = l\omega_3 - n\omega_1,$$
$$\dot{n} = m\omega_1 - l\omega_2.$$

Thus

$$\dot{\Omega} = 2(l\dot{\omega}_1 + m\dot{\omega}_2 + n\dot{\omega}_3)$$
$$= 2l\frac{B-C}{A}\omega_2\omega_3 + 2m\frac{C-A}{B}\omega_3\omega_1 + 2n\frac{A-B}{C}\omega_1\omega_2.$$

Hence a body moving with S, whose moments of inertia about the fixed centre are all equal, say to A_1, must be acted on by a couple, whose components are

$$2A_1\frac{B-C}{A}\omega_2\omega_3,\quad 2A_1\frac{C-A}{B}\omega_3\omega_1,\quad 2A_1\frac{A-B}{C}\omega_1\omega_2.$$

The plane of this couple contains the two directions $(A\omega_1, B\omega_2, C\omega_3)$, $(A'\omega_1, B'\omega_2, C'\omega_3)$ that is, the invariable

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lines in the two Poinsot motions. Its components are therefore proportional to

$$(BC' - B'C)\,\omega_2\omega_3,\ (CA' - C'A)\,\omega_3\omega_1,\ (AB' - A'B)\,\omega_1\omega_2,$$

and in fact it is easily found that they are equal, respectively, to these quantities multiplied by

$$2A_1/ABC,$$

in Dr. Routh's notation.

But these are the moments of a force, whose components are

$$(\lambda A\omega_1,\ \lambda B\omega_2,\ \lambda C\omega_3),$$

acting at a point in the invariable line of the second motion, whose coordinates are

$$(\mu A'\omega_1,\ \mu B'\omega_2,\ \mu C'\omega_3),$$

if $$\lambda\mu = -2A_1/ABC.$$

This relation is satisfied if we put

$$\lambda = -g/G,\ \mu = h_1/G'.$$

The force is then constant $(-g)$, in a fixed direction, namely, that of the invariable line in the first motion, and applied at a fixed point on the invariable line in the second motion.

Thus the motion of S is exactly that of a heavy body or top hung up by a point about which all the moments of inertia are equal, the first invariable line being the upward vertical, and the second passing through the centre of inertia of the top.

Let us understand by *associated* Poinsot motions those in which confocal conicoids roll upon parallel planes, the component angular velocities about axes parallel to those planes being the same for all when they reach the same position. Thus any Poinsot motion can be derived from any of its associated motions by the addition of a uniform rotation about the invariable line.

For the second of the two Poinsot motions hitherto discussed substitute any of its associates, that is, compound with the former motion of S a constant angular velocity, say σ, about the second invariable line.

The component angular velocities of S are therefore now

$$2\omega_1 + \sigma A'\omega_1/G',\ 2\omega_2 + \sigma B'\omega_2/G',\ 2\omega_3 + \sigma C'\omega_3/G',$$

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while

$$\dot{l} = n\omega_2 (1 + \sigma B'/G') - m\omega_3 (1 + \sigma C'/G'), \text{ etc.,}$$

and

$$\Omega = 2 (l\omega_1 + m\omega_2 + n\omega_3) + \frac{\sigma}{G'} (A'l\omega_1 + B'm\omega_2 + C'n\omega_3).$$

Thus

$$\dot{\Omega} = 2l \frac{B - C}{A} \omega_2\omega_3 + 2m \frac{C - A}{B} \omega_3\omega_1 + 2n \frac{A - B}{C} \omega_1\omega_2$$

$$+ \frac{2\sigma}{G'} \{l\omega_2\omega_3 (C' - B') + m\omega_3\omega_1 (A' - C') + n\omega_1\omega_2 (B' - A')\}.$$

As before, this expression vanishes if (l, m, n) coincides with the second invariable line. The angular velocity about that line has the constant value $2T'/G' + \sigma$. Let us write n_1 for this.

Now let (l, m, n), (l', m', n') be *any* two directions at right angles in the second invariable plane, moving with S; we have

$$G'l' = nB'\omega_2 - mC'\omega_3,$$
$$G'm' = lC'\omega_3 - nA'\omega_1,$$
$$G'n' = mA'\omega_1 - lB'\omega_2,$$

the rotation from (l, m, n) to (l', m', n') being in the positive direction.

Then the new terms in the expression for $\dot{\Omega}$ are equal to

$$2\sigma (l'\omega_1 + m'\omega_2 + n'\omega_3),$$

say $$\sigma\Omega',$$

Ω' being the angular velocity of S about the axis (l', m', n').

But now if $$T'/G' = C_1 n_1/2A_1,$$

$$\sigma = n_1 (A_1 - C_1)/A_1.$$

Thus the equations of motion of S coincide with those of a top whose moment of inertia is C_1 about the second invariable line, which is its axis of figure, and A_1 about any axis perpendicular thereto, the applied force being the same as before.

If then the two are started from the same position and with the same velocities, they will always be found in the same position.

In order that the positions at any particular moment may be the same, we must have, at that moment,

$$AA'\omega_1^2 + BB'\omega_2^2 + CC'\omega_3^2 = GG' \cos\theta.$$

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In order that the velocities may be the same, we must have

$$n_1 = 2T/G + \sigma,$$

these being the components about the second invariable line

$$E_1/A_1 + (A_1 - C_1)n_1 \cos\theta, \; A_1 = 2T/G + \sigma\cos\theta,$$

these being the components about the first invariable line; lastly

$$-GG'\dot{\theta}\sin\theta = 2AA'\omega_1\dot{\omega}_1 + 2BB'\omega_2\dot{\omega}_2 + 2CC'\omega_3\dot{\omega}_3$$
$$= 2\omega_1\omega_2\omega_3 i(B-C)(C-A)(A-B).$$

To these must be added the equations

$$\sigma = n_1(A_1 - C_1)A_1,$$
$$gh_1/2A_1 = GG'/ABC,$$

which involve the proportions of the top.

When these proportions and the initial circumstances of the top are given, the foregoing six equations become, by substitution of the real finite value given for σ by the last but one, just the system treated by Dr. Routh and shewn by him to yield a relevant set of values for $A, B, C, T, G, \omega_1, \omega_2, \omega_3$.

The theorem of Jacobi may then be stated as follows:—

Take any polhode and let E, E_1 be the two different Poinsot motions belonging to it, the angular velocities in the two being opposite and equal. Let E_1 be a Poinsot motion associated with E. Then if a body performs the motion E in a fixed space S_0, and the motion E_1 at the same time in a moving space S, the motion that S must have relatively to S_0 is exactly that of a top, and by proper choice of the constants of the Poinsot motions, it can be made to represent any motion of any top.

CYCLIC NUMBERS.

By Leonard E. Dickson, M.A.

I. *Numbers with Cyclic Multiples.*

A NUMBER of D digits written to the base N, $(A_1A_2A_3\ldots A_{D-1}A_D)$, will be said to *possess cyclic multiples* when, if we multiply it by D different integers $E_1 = 1, E_2, E_3, \ldots, E_D$, each product contains the digits of the given number and *in the same cyclic order*. Thus 142,857 to the base 10; also 1463 to the base 8.

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By definition we have the $D-1$ symbolic equations:

$$(A_2A_3A_4\ldots\ldots A_{D-1}A_DA_1) = B_2\ (A_1A_2A_3A_4\ldots A_{D-1}A_D),$$

$$(A_3A_4\ldots\ldots A_{D-1}A_DA_1A_2) = B_3\ (\quad\quad\quad

$$\Delta_1 = \begin{vmatrix} B_2N^{D-2}-N^{D-1}, & B_2N^{D-3}-N^{D-2}, \ldots, & B_2N^{D-k}-N^{D-k+1}, \ldots, & B_2-N \\ B_3N^{D-2}-1, & B_3N^{D-3}-N^{D-1}, \ldots, & B_3N^{D-k}-N^{D-k+2}, \ldots, & B_3-N^2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ B_KN^{D-2}-N^{K-3}, & B_KN^{D-3}-N^{K-4}, \ldots, & B_KN^{D-k}-N^{D-k+K-1}, \ldots, & B_K-N^{K-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ B_DN^{D-2}-N^{D-3}, & B_DN^{D-3}-N^{D-4}, \ldots, & B_DN^{D-k}-N^{D-k-1}, \ldots, & B_D-N^{D-1} \end{vmatrix}.$$

From each column subtract the following column multiplied by N; therefore

$$\Delta_1 = \begin{vmatrix} 0, & 0, & 0, \ldots, & 0, & B_2-N \\ N^D-1, & 0, & 0, \ldots, & 0, & B_3-N^2 \\ 0, & N^D-1, & 0, \ldots, & 0, & B_4-N^3 \\ 0, & 0, & N^D-1, \ldots, & 0, & B_5-N^4 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0, & 0, & 0, \ldots, & N^D-1, & B_D-N^{D-1} \end{vmatrix}.$$

Therefore $\Delta_1 = (N^D-1)^{D-2}(N-B_2)$.

$$\Delta_2 = -A_1 \begin{vmatrix} B_2N^{D-1}-1, & B_2N^{D-3}-N^{D-2}, \ldots, & B_2N^{D-k}-N^{D-k+1}, \ldots, & B_2-N \\ B_2N^{D-1}-N, & B_2N^{D-3}-N^{D-1}, \ldots, & B_2N^{D-k}-N^{D-k+2}, \ldots, & B_2-N^2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ B_KN^{D-1}-N^{K-2}, & B_KN^{D-3}-N^{K-4}, \ldots, & B_KN^{D-k}-N^{D-k+K-1}, \ldots, & B_K-N^{k-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ B_DN^{D-1}-N^{D-2}, & B_DN^{D-3}-N^{D-4}, \ldots, & B_DN^{D-k}-N^{D-k-1}, \ldots, & B_D-N_{D-1} \end{vmatrix}.$$

From each column subtract the following column multiplied by N, except the second which is multiplied by N^2. Therefore

$$\Delta_2 = -A_1 \begin{vmatrix} N^D-1 & 0 & 0 & 0 & B_2-N \\ N^{D+1}-N, & 0 & 0 & 0 & B_3-N^2 \\ 0 & N^D-1, & 0 & 0 & B_4-N^3 \\ 0 & 0 & N^D-1, & 0 & B_5-N^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & N^D-1, & B_D-N^{D-1} \end{vmatrix}.$$

Therefore

$$\Delta_2 = -A_1(N^D-1)^{D-3} \begin{vmatrix} N^D-1 & B_2-N \\ N^{D+1}-N, & B_3-N^2 \end{vmatrix}$$

$$= A_1(N^D-1)^{D-2}(NB_2-B_3).$$

Similarly

$$\Delta_3 = A_1(N^D-1)^{D-2}(NB_3-B_4),$$

$$\Delta_4 = A_1(N^D-1)^{D-2}(NB_4-B_5),$$

$$\cdots\cdots\cdots\cdots,$$

$$\Delta_K = A_1(N^D-1)^{D-2}(NB_K-B_{K+1}),$$

$$\cdots\cdots\cdots\cdots,$$

$$\Delta_D = A_1(N^D-1)^{D-2}(NB_D-1).$$

Therefore

$$A_2 = \frac{A_2(NB_2-B_3)}{N-B_3}; \quad A_3 = \frac{A_1(NB_3-B_4)}{N-B_2},$$

$$A_4 = \frac{A_1(NB_4-B_5)}{N-B_3}; \ldots, A_D = \frac{A_1(NB_D-1)}{N-B_2}.$$

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Substituting these values in the expansion

$$A_1.N^{D-1} + A_2.N^{D-2} + A_3.N^{D-3} + \ldots + A_{D-1}.N + A_D$$

of the number $(A_1A_2A_3\ldots A_D)$, it reduces to $\frac{A_1(N^D-1)}{N-B_2}$.

Hence every number of D digits written to the base N must be of this form if it possesses cyclic multiples.

Let the highest common factor of A_1 and $N-B_2$ be cancelled out, the form becoming $\frac{A(N^D-1)}{P}$, where A and P are relatively prime. A number of this form is an integer only when P is prime to N, and D is a multiple of d, the exponent to which N belongs modulo P. If the number $\frac{A(N^d-1)}{P}$ be cyclic, it will be called *mono-cyclic* in distinction from the number $\frac{A(N^{md}-1)}{P}$, composed of the group of digits given by the former repeated m times and called *poly-cyclic.*

An integral number of D digits of the form $\frac{A(N^D-1)}{P}$ will have its digits all alike if, and only if, P is a divisor of $M=N-1$.

THEOREM. *The cyclic multipliers of any mono-cyclic number are the least residues modulo P of $N, N^2, N^3, \ldots, N^D$.*

N^D-1 written to the base N is $(MMM\ldots)$ to D digits, where $M=N-1$.

$$A(MM\ldots M) = A\{(100\ldots0)-1\} = ((A-1)MM\ldots(N-A)).$$

Divide by P and call the quotients $A_1, A_2, \ldots, A_D$ and the corresponding remainders $R_1, R_2, \ldots, R_D$, where R_D alone $=0$.

Then

$$\begin{aligned}
(A-1)N + M &= A_1 P + R_1,\\
R_1N + M &= A_2 P + R_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
R_{K-1}N + M &= A_K P + R_K,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
R_{D-1}N + (N-A) &= A_D P.
\end{aligned}$$

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Now B_K is the multiplier which brings the digit A_K of the cyclic number into the first place of the product. But $B_K A(MM...M) = ((B_K A - 1)MM...(N - B_K A))$. Hence must $B_K A - 1 = R_{K-1}$, as shown by the general equation above. Thus the cyclic multipliers are

$$\frac{R_1 + 1}{A}, \frac{R_2 + 1}{A}, ..., \frac{R_{D-1} + 1}{A}, 1.$$

Since $R_1, R_2, R_3, ...$, are the remainders left on dividing $(A-1)M$, $(A-1)MM$, $(A-1)MMM$, ..., respectively by P, they are the least residues modulo P of $AN-1$, AN^2-1, AN^3-1, ..., AN^D-1. Hence the cyclic multipliers are the quotients of the least residues of AN, AN^2, AN^3, ..., AN^D modulo P by A. If n_K is the least residue modulo P of N^K, the least residue of An_K is that of AN^K. But An_K must be $< P$, and hence its own least residue modulo P. For, if $> P$, suppose $An_K = lP + n'_K$. Then $l < A$, since $P > n_K$. But the least residue n'_K of An_K is divisible by A, the quotient being a cyclic multiplier. Hence, A being prime to P, l is divisible by A, therefore $l = 0$. Hence the cyclic multipliers are $n_1, n_2, ..., n_D$.

This theorem admits of direct proof. Since neither N^K nor $N^K - 1$ is divisible by P, when $K < D$, the greatest integer in $\frac{N^K - 1}{P}$ equals that in $\frac{N^K}{P}$. Hence the least residue modulo P of N^K is

$$N^K - (A_1 N^{K-1} + A_2 N^{K-2} + ... + A_{K-1} N + A_K)\frac{P}{A}.$$

Now $$\alpha\left\{\frac{P}{A}(A_1 A_2 A_3 ... A_D)\right\} = \alpha\{(1000...0) - 1\}.$$

In our case

$$\alpha = A_1 N^{K-1} + ... + A_{K-1} N + A_K = (A_1 A_2 A_3 ... A_K);$$

hence the product is

$$\{(A_1 A_2 A_3 ... A_K 000...0), \text{ of } D + K \text{ digits}, -(A_1 A_2 A_3 ... A_K)\}.$$

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But this is to be subtracted from

$$N^K(A_1A_2A_3\ldots A_D) = (A_1A_2A_3\ldots A_D00\ldots0),$$

of $D+K$ digits.

The result is

$$\{(A_{K+1}A_{K+2}\ldots A_D00\ldots0), \text{ of } D \text{ digits}, + (A_1A_2\ldots A_K)\}$$
$$= (A_{K+1}A_{K+2}\ldots A_DA_1A_2\ldots A_K).$$

The following theorem follows from the first proof of proof of the last theorem, if we make $A=1$.

Every number $\frac{N^D-1}{P}$, *where* P *is prime to* N, *is mono-cyclic if* N *belongs to the exponent* D *modulo* P.

Hence every integer of the form $\frac{N^D-1}{P}$, when written to the base N, is either mono-cyclic or poly-cyclic (including the case when all its digits are alike).

I confine myself henceforth to cyclic numbers which are integers, discarding those cyclic "decimals," such as

$$\cdot0588235294117647$$

to the base 10, which have one or more zeros to the left of the significant digits. Hence must $P<N$ and the first digit $A_1<\frac{N}{D}$. Further, the digits A_s must all be different. For in the products by B_s each digit must appear in the first place once and only once. But, these multipliers being all different, these first digits must also.

As a notation for a cyclic number, I use

$$\left[\begin{matrix} A_1, A_2, A_3, \ldots A_D \\ 1, B_2, B_3, \ldots, B_D \end{matrix}\right]^{N}_{P}. \qquad \text{Example } \left[\begin{matrix} 1\ 3\ \text{XI}\ 6\ 5 \\ 1, 3, 9, 5, 4 \end{matrix}\right]^{14}_{11}.$$

The great majority of integers of the form $A\frac{N^D-1}{P}$,

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where $A > 1$, and N belongs to the exponent D mod. P, will possess less than D cyclic multiples. Such cyclics will be be called *impure*.

Thus, for $A = 2$, $\begin{bmatrix} 2, & \text{XI}, & 7 \\ 1, & 4, & \end{bmatrix} \begin{matrix} 13 \\ 9 \end{matrix}$; $\begin{bmatrix} 2 & \text{XX} & \text{XI} & \text{XVI} \\ & 1, & 7, & 4 \end{bmatrix} \begin{matrix} 22 \\ 15 \end{matrix}$.

To find for a given base N all pure mono-cyclic numbers given by $A > 1$, every value of P less than N, prime to N and not a divisor of $N-1$ is to be taken as modulus. If the least residue of any power ($< D$) of N is $> P/A$, such a combination of values for P and A must be discarded. For $N < 19$, no pure cyclic ($A > 1$) exists. For $N = 19$ we find one, $\begin{bmatrix} 2 & \text{X} \\ 1, & 4 \end{bmatrix} 15$ for $A = 2$.

It is best to start with a given modulus P, and investigate the infinity of values of N by considering their least residues modulo P. Such a residue n must be prime to P and, in particular, cannot be 2; for, P being odd, one power of 2 lies between $P/2$ and P. Taking for n each integer, except 2, which is $< P/A$ and prime to P, we discard everyone, the least residue of any power of which is $> P/A$.

The following list of forms gives all the values of N for which pure mono-cyclics ($A > 1$) exist for $P \leqq 60$; m being an integer $\geqq 1$, N may be =

$15m+4$, $A=2$; $21m+8$, $A=2$; $33m+10$, $A=2$;

$35m+6$, $A=2, 3, 4$; $35m+11$, $A=2$; $35m+16$, $A=2$;

$38m+7, +11$, $A=3$; $39m+14$, $A=2$; $40m+9, +11$, $A=3$;

$45m+17, +19$, $A=2$; $48m+7$, $A=5$; $51m+4, +13, +16$, $A=2$;

$55m+21$, $A=2$; $56m+13, +15$, $A=3$; $57m+20$, $A=2$.

Hence the following list of 19 pure mono-cyclics ($A > 1$) is complete for bases $\leqq 63$;

$\begin{bmatrix} 2 & \text{X} \\ 1, & 4 \end{bmatrix} \begin{matrix} 19 \\ 15 \end{matrix}$; $\begin{bmatrix} 2 & \text{XXII} \\ 1, & 8 \end{bmatrix} \begin{matrix} 29 \\ 21 \end{matrix}$; $\begin{bmatrix} 4 & \text{XVIII} \\ 1, & 4 \end{bmatrix} \begin{matrix} 34 \\ 15 \end{matrix}$; $\begin{bmatrix} 2 & \text{XIV} \\ 1, & 6 \end{bmatrix} \begin{matrix} 41 \\ 35 \end{matrix}$;

$\begin{bmatrix} 3 & \text{XXI} \\ 1, & 6 \end{bmatrix} \begin{matrix} 41 \\ 35 \end{matrix}$; $\begin{bmatrix} 4 & \text{XXVIII} \\ 1, & 6 \end{bmatrix} \begin{matrix} 41 \\ 35 \end{matrix}$; $\begin{bmatrix} 2 & \text{XXVI} \\ 1, & \text{X} \end{bmatrix} \begin{matrix} 43 \\ 33 \end{matrix}$;

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$$\begin{bmatrix} 3 & \text{XXIV} & \text{XXXIX} \\ 1, & 7, & \text{XI} \end{bmatrix}^{45}_{35}; \quad \begin{bmatrix} 2 & \text{XXVIII} & \text{XLII} \\ 1, & \text{XI}, & \text{XVI} \end{bmatrix}^{46}_{35};$$

$$\begin{bmatrix} 3 & \text{XXXIII} \\ 1, & 9 \end{bmatrix}^{49}_{40}; \quad \begin{bmatrix} 6 & \text{XXVI} \\ 1, & 4 \end{bmatrix}^{49}_{15}; \quad \begin{bmatrix} 3 & \text{XLII} & \text{XXVII} \\ 1, & 7, & \text{XI} \end{bmatrix}^{49}_{38};$$

$$\begin{bmatrix} 4 & \text{XXXIV} \\ 1, & 8 \end{bmatrix}^{50}_{21}; \quad \begin{bmatrix} 3 & \text{XLII} \\ 1, & \text{XI} \end{bmatrix}^{51}_{40}; \quad \begin{bmatrix} 2 & \text{XLVI} & \text{XXXII} \\ 1, & \text{XI}, & \text{XIII} \end{bmatrix}^{51}_{35};$$

$$\begin{bmatrix} 2 & \text{XXXVIII} \\ 1, & \text{XIV} \end{bmatrix}^{53}_{39}; \quad \begin{bmatrix} 5 & \text{XL} \\ 1, & 7 \end{bmatrix}^{55}_{48};$$

$$\begin{bmatrix} 2 & 8 & \text{XXXIV} & \text{XXVIII} \\ 1, & 4, & \text{XVI}, & \text{XIII} \end{bmatrix}^{55}_{51}; \quad \begin{bmatrix} 2 & \text{XLVI} & \text{LII} & \text{XXII} \\ 1, & 8, & \text{XVII}, & \text{XIX} \end{bmatrix}^{62}_{45}.$$

The only pure mono-cyclic number belonging to the base 10 is 142857. There are 116 pure mono-cyclics for bases $\leqq 25$, all but one given by $A = 1$.

If the number of digits of a mono-cyclic number (given by $A = 1$) be even, the K^{th} and $(D/2 + K)^{\text{th}}$ digits will constitute a pair of *opposites*.

The necessary and sufficient condition that the sum of each pair of opposite digits shall be $N - 1$ is $\{N^{D/2}\} = -1$, where $\{x\}$ denotes the least residue of x modulo P.

If $\{N^{D/2}\} = -1$, then $\{N^{D/2+K}\} = P - \{N^K\}$.

But $$A^K = \frac{R_{K-1}N + M - R_K}{P} = \frac{N\{N^{K-1}\} - \{N^K\}}{P},$$

$$A_{D/2+K} = \frac{N\{N^{D/2+K-1}\} - \{N^{D/2+K}\}}{P}$$

$$= \frac{PN - N\{N^{K-1}\} - P + \{N^K\}}{P}.$$

Therefore $$A_K + A_{D/2+K} = \frac{P(N-1)}{P} = N - 1.$$

Inversely, let $A_K + A_{D/2+K} = N - 1$, $(K = 1, 2, \ldots, D)$.

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Now B_K times the cyclic number $(A_1A_2\ldots A_{K-1})$ gives $(A_KA_{K+1}\ldots A_{2K-1})$; $B_{D/2+K}$ times it gives

$$(A_{D/2+K}A_{D/2+K+1}\ldots A_{D/2+K-1}).$$

Applying the hypothesis, the sum of the two products equals $(MM\ldots M)$ or $P(A_1A_2\ldots A_D)$. Therefore $B_K+B_{D/2+K}=P$, *i.e.* $\{N^K\}+\{N^{D/2+K}\}=P$. Therefore $\{N^{D/2}\}=-1$.

Whenever the sum of each pair of opposite digits $=N-1$, the sum of each pair of opposite multipliers $=P$, and inversely.

If $P=p^K$, where p is a prime number >2, $\{N^{D/2}\}=-1$, $(N^D-1)=(N^{D/2}-1)(N^{D/2}+1)$ is divisible by p^K. The first factor is divisible by p^{K-1} at most, since N belongs to the exponent D modulo p^K. Hence, $N^{D/2}+1$ is divisible by p. Therefore $N^{D/2}-1$ is prime to $p>2$, and $N^{D/2}+1$ is divisible by p^K. The same holds when $P=2p^K$.

If $P=4$, N must be odd. But $N\neq 4n+1$, since $N-1$ would be divisible by $P=4$. Therefore $\{N\}=-1$.

If N be a primitive root of the modulus P, the cyclic number may be called *primitive*. Hence, in every primitive mono-cyclic, the sum of each pair of opposite digits $=N-1$, and the sum of each pair of opposite multipliers $=P$.

Such cyclic numbers (including all primitive ones) may be well represented by marking the points of division of a circumference into D equal parts with the digits of the cyclic number on the outside and with the corresponding multipliers on the inside. Then will the sum of opposite digits $=N-1$ and the sum of opposite multipliers $=P$. (Figs. 45, 46).

No cyclic number given by $A>1$ can have this property, since each multiplier must be $<P/A$ and hence $<P/2$.

Adding the D products of a cyclic number $(A_1A_2A_3\ldots A)_D$ by the multipliers $B_1=1, B_2, B_3, \ldots, B_D$, we derive the equation

$$P(A_1+A_2+A_3+\ldots+A_D)=A(N-1)(B_1+B_2+B_3+\ldots+B_D).$$

By Gauss, *D.A.*, Art. 79, the sum of the D cyclic multipliers, $1+\{N\}+\{N^2\}+\ldots+\{N^D\}$, is divisible by P.

II.—*Numbers with Cyclic Quotients.*

A number of D digits written to the base N is said to possess *cyclic quotients* when, if we divide it by D different

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integers $C_1 = 1$, C_2, C_3, ..., C_D, each quotient contains the digits of the given number and in the same cyclic order. Thus, 512 to the base 9.

We have $D-1$ symbolic equations:

$$A_1A_2A_3\ldots A_D = C_K(A_KA_{K+1}A_{K+2}\ldots A_{K-1}),\ (K=2, 3, \ldots, D).$$

Expanding these and solving for A_1, A_2, ..., A_{D-1} in terms of A_D, N, C_2, C_3, ..., C_D, we find

$$A_1 = \frac{A_DC_D(C_2N-1)}{C_2(N-C_D)},\quad A_2 = \frac{A_DC_D(C_3N-C_2)}{C_2C_3(N-C_D)},$$

$$A_3 = \frac{A_DC_D(C_4N-C_3)}{C_3C_4(N-C_D)},\ \ldots,\ A_{D-1} = \frac{A_DC_D(C_DN-C_{D-1})}{C_{D-1}C_D(N-C_D)}.$$

Substituting these values in the expansion of the number $(A_1A_2A_3\ldots A_{D-1})$, it reduces to $\dfrac{A_{D-1}C_{D-1}(N_{D-1}-1)}{N-C_{D-1}}$.

Let the greatest factor common to A_DC_D and $N-C_D$ be cancelled out, the form becoming $\dfrac{E(N^D-1)}{Q}$.

THEOREM. The cyclic quotients are all multiples of the number $\dfrac{N_{D-1}-1}{Q}$, or, E is divisible by C_2, C_3, ..., C_{D-1}.

Since $A_1 = \dfrac{E(C_2N-1)}{C_2Q}$, $A_1Q = EN - \dfrac{E}{C_2}$.

Similarly
$$A_2Q = \frac{EN}{C_2} - \frac{E}{C_3},$$
$$A_3Q = \frac{EN}{C_3} - \frac{E}{C_4},\ \&c.$$

Let the division of $E(MM\ldots M)$ by Q give the quotients A_1, A_2, ..., A_{D-1} and the remainders R_1, R_2, ..., $R_{D-1} = 0$, respectively. Since E/C_K is an integer, say G_K, the quotient of $(A_1A_2\ldots A_D)$ by C_K is

$$\frac{G_K\,(MM\ldots M)}{Q} = \frac{((G_K-1)\,MM\ldots(N-G_K))}{Q}.$$

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Therefore

$$R_{K-1} = G_K - 1, \text{ or } C_K = \frac{E}{R_{K-1}+1} \quad (K=2, 3, \ldots, D);$$

while $$C_1 = 1.$$

But $R_1, R_2, \ldots, R_{D-1}$ are the least residues modulo Q of $EN-1, EN^2-1, \ldots, EN^{D-1}-1$. Hence the cyclic divisors $C_2, C_3, \ldots, C_D$ are the quotients of E by the least residues modulo Q of $EN, EN^2, \ldots, EN^{D-1}$.

From $C_K = \dfrac{E}{\{EN^{K-1}\}}$ we find, since $E = C_K . G_K$, where G_K is an integer, $\{G_K C_K N^{K-1}\} = G_K$.

Therefore $$\{C_K N^{K-1}\} = 1 \text{ or } C_K = \{N^{D-K+1}\}.$$

Hence the cyclic divisors $C_1, C_2, C_3, \ldots, C_D$ are the least residues modulo Q of $N^D, N^{D-1}, N^{D-2}, \ldots, N$ respectively.

The condition that there exists for a given base N and modulus Q a number possessing D cyclic quotients (D being the exponent to which N belongs modulo Q), is that the least common multiple of the least residues of the first D powers of N be $< Q$.

From every number of *two* digits possessing cyclic multiples we may derive one possessing cyclic quotients.

Examples of numbers with cyclic quotients:

$$\left[\begin{matrix} 6\ 3\ 1 \\ 1,\ 2,\ 4 \end{matrix}\right] \frac{11}{7}, \quad \left[\begin{matrix} 9\ 2\ 4 \\ 1,\ 4,\ 2 \end{matrix}\right] \frac{16}{7}, \quad \left[\begin{matrix} \text{XI}\ 1\ 3 \\ 1,\ 9,\ 3 \end{matrix}\right] \frac{16}{13}, \quad \left[\begin{matrix} 9\ 1\ 2\ 4 \\ 1,\ 8,\ 4,\ 2 \end{matrix}\right] \frac{17}{15},$$

$$\left[\begin{matrix} \text{XII}\ 6\ 3\ 1 \\ 1,\ 2,\ 4,\ 8 \end{matrix}\right] \frac{23}{15}, \quad \left[\begin{matrix} \text{XIX}\ 1\ 4 \\ 1,\ \text{XVI},\ 4 \end{matrix}\right] \frac{25}{21},$$

$$\left[\begin{matrix} \text{XXIV}\ \text{XII}\ 6\ 3\ 1 \\ 1,\ 2,\ 4,\ 8,\ \text{XVI} \end{matrix}\right] \frac{47}{31}.$$

The University of Chicago,
May 3, 1895.

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ON A LITTLE CIRCLE SPHERICAL TRIANGLE.

By R. C. Hudson, B.A., late Scholar of St. John's College, Cambridge.

1. LET ABC be a little-circle triangle, the latitudes of whose sides are α, β, γ from the great circles parallel to them. Let DEF be the corresponding great-circle triangle, so that the angles D, E, F are respectively equal to A, B, C. (Fig. 47).

Draw BB_1, CC_2 perpendicular to EF; CC_1, AA_2 to FD; and AA_1, BB_2 to DE.

Let $BC=a$, $CA=b$, $AB=c$; $DA_1=a_1$, $DA_2=a_2$; $EB_1=b_1$, $EB_2=b_2$; $FC_1=c_1$, $FC_2=c_2$. Then $BB_1=CC_2=\alpha$, $CC_1=AA_2=\beta$, $AA_1=BB_2=\gamma$; and $B_1C_2=a$, $C_1A_2=b$, $A_1B_2=c$.

Join AD. Let $ADA_1=\phi$: then $ADA_2=A-\phi$,

$$\frac{\sin(A-\phi)}{\sin\phi}=\frac{\sin\beta}{\sin AD}\cdot\frac{\sin AD}{\sin\gamma}=\frac{\sin\beta}{\sin\gamma};$$

hence
$$\cot\phi=\frac{\sin\beta+\sin\gamma\cos A}{\sin\gamma\sin A},$$

and
$$\tan\phi=\frac{\tan\gamma}{\sin a_1}.$$

Hence

$$\sin a_1=\frac{\sin\beta+\sin\gamma\cos A}{\cos\gamma\sin A},\quad \sin a_2=\frac{\sin\gamma+\sin\beta\cos A}{\cos\beta\sin A}\ldots\ldots(1),$$

$$\sin b_1=\frac{\sin\gamma+\sin\alpha\cos B}{\cos\alpha\sin B},\quad \&c.$$

and
$$\cos A=-\cos B\cos C+\sin B\sin C\cos(a+b_1+c_2)\ldots(2),$$

or

$$\cos A=-\cos B\cos C+\sin B\sin C\cos\left\{a+\sin^{-1}\left(\frac{\sin\gamma+\sin\alpha\cos B}{\cos\alpha\sin B}\right)\right.$$
$$\left.+\sin^{-1}\left(\frac{\sin B+\sin\alpha\cos C}{\cos\alpha\sin C}\right)\right\}\ldots\ldots\ldots(3).$$

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A corresponding expression for $\cos a$ in terms of b, c, A and the latitudes, which may be obtained by solid geometry, is given after; but we will first make two easy applications of (2) and (3).

2. From (2), putting $\tan\frac{1}{2}a = x$, $\cos A + \cos B \cos C = a$, $\sin B \sin C = b$, $\cos(b_1 + c_2) = h$, $\sin(b_1 + c_2) = \dot{h}$, we have

$$a = b\left(\frac{1-x^2}{1+x^2}.h - \frac{2x}{1+x^2}.\dot{h}\right),$$

whence

$$\tan\tfrac{1}{2}a = \frac{2N - \sin B \sin C \sin(b_1 + c_2)}{\cos A + \cos B \cos C + \sin B \sin C \cos(b_1 + c_2)} \ldots(4),$$

where N has its ordinary meaning, namely

$$N = \sqrt{\{\sin E \sin(A-E) \sin(B-E) \sin(C-E)\}} \ldots(5),$$

where $$2E = A + B + C - \pi.$$

3. Case $C = \frac{1}{2}\pi$, $\beta = 0$, $\gamma = 0$; or right-angled triangle, with one side a little circle (fig. 48). Here

$$\cos A = \sin B \cos\left\{a + \sin^{-1}\left(\frac{\tan\alpha}{\tan B}\right)\right\},$$

$$\cos B = \sin A \cos(b + \alpha),$$

$$\cot A \cot B = \cos\left\{c + \sin^{-1}\left(\frac{\sin\alpha}{\sin B}\right)\right\};$$

or

$$u \cos A = p\sqrt{(u^2 - \cos^2 B)} - \dot{p}\dot{u}\cos B \ldots\ldots(6),$$

$$\cos B = s \sin A \ldots\ldots(7),$$

$$\cot A \cos B = r\sqrt{(u^2 - \cos^2 B)} - \dot{r}\dot{u} \ldots\ldots(8),$$

where $p = \cos a$, $\dot{p} = \sin a$, $q = \cos b$, $\dot{q} = \sin b$, $r = \cos c$, $\dot{r} = \sin c$, $s = \cos(b + \alpha)$, $\dot{s} = \sin(b + \alpha)$; $u = \cos\alpha$, $\dot{u} = \sin\alpha$, $v = \cos\beta$, $\dot{v} = \sin\beta$, $w = \cos\gamma$, $\dot{w} = \sin\gamma$ ...(9).

Putting $\tan A = x$, we have from (6) and (7)

$$x = \frac{\dot{p}u}{\dot{p}su - \dot{s}u} \text{ or } \frac{-\dot{p}u}{\dot{p}su + \dot{s}u},$$

where the second value, not holding for $\alpha = 0$, must be rejected.

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Hence

$$\tan A = \frac{\dot{p}u}{\dot{p}su - \dot{s}u} \quad \text{.....(10).}$$

Again, from (7) and (8), neglecting a value which does not hold when $\alpha = 0$, we have

$$\cos A = \frac{\dot{s}r - \dot{u}}{\dot{s}r} \quad \text{... .....(11).}$$

From (7) and (11),

$$\sqrt{(u^2 - \cos^2 B)} = \sqrt{(u^2 - s^2 + s^2 \cos^2 A)} = \pm \frac{\dot{s} - r\dot{u}}{\dot{r}},$$

where the upper sign must be taken, since, when $\alpha = 0$, $\sin B = \frac{\dot{s}}{\dot{r}}$. From this, (11), (7), and (6),

$$\sin A = \frac{(\dot{u} - \dot{s}r)\, u + ps\,(\dot{s} - r\dot{u})}{\dot{p}s\dot{r}u} \quad \text{.....(12).}$$

Equating the value of $\tan A$ from (12) and (11), with (10), we have for the relation between the sides

$$psu + \dot{s}\dot{u} - r = 0.$$

Substituting this relation in (12), we have

$$\sin A = \frac{\dot{p}u}{\dot{r}},$$

that is,

$$\cos c = \cos a \cos(b + \alpha) \cos\alpha + \sin(b + \alpha) \sin\alpha \text{...(13),}$$

$$\sin A = \frac{\sin a \cos\alpha}{\sin c} \quad \text{.....(14),}$$

$$\cos A = \frac{\sin(b + \alpha) \cos c - \sin\alpha}{\cos(b + \alpha) \sin c} \quad \text{.....(15),}$$

$$\cos B = \frac{\sin a \cos(b + \alpha) \cos\alpha}{\sin c} \quad \text{.....(16).}$$

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4. Let ABC be the little-circle triangle, DEF the corresponding great-circle triangle. Let OD be taken for the axis OX, and ODE for the plane XOY. Let x_3, y_3, z_3 be the coordinates of B, and x_2, y_2, z_2 of C (fig. 49).

Let the little-circle plane AB cut OZ in L, the plane ZOX in LM, ZOY in LT, and circle DF in K. Let plane $ODKF$ cut LT in S, OY in H, and circle DE in Q. Let the plane $DOGP$ be drawn perpendicular to plane $ODKF$ intersecting ZOY in OG, SH produced in G, and circle CAQ produced in P, so that the points O, G, H, S, N, L are the projections on the plane ZOY of D, P, Q, A, K, M. Draw CW perpendicular to HQ, CV perpendicular to plane XOY, and join WV, which is therefore parallel to OHY.

Then L is the centre of the little-circle AB, and with the notation of (9), $LA = Rw$, R being the radius of the sphere; and G is centre of the little-circle AC, and $GA = Rv$,

$$OL = R\dot{w},\ LON = \tfrac{1}{2}\pi - A,\ ONL = A;$$

therefore
$$NL = \frac{R\dot{w}\cos A}{\sin A},\quad ON = \frac{R\dot{w}}{\sin A}.$$

$$OG = R\dot{v},\ HOG = \tfrac{1}{2}\pi - A,\ OHG = A;$$

therefore
$$HG = \frac{R\dot{v}\cos A}{\sin A},\quad OH = \frac{R\dot{v}}{\sin A},$$

$$\sin ALM = \frac{SL}{LA} = \frac{HO + NL}{LA} = \frac{\dot{v} + \dot{w}\cos A}{w\sin A} \ldots (17),$$

$$\sin AGP = \frac{SG}{GA} = \frac{NO + HG}{GA} = \frac{\dot{w} + \dot{v}\cos A}{v\sin A} \ldots (18),$$

$$CWV = A,\ CW = GC.\sin(CA + AP) - HG$$

$$= Rv\sin(b + AP) - \frac{R\dot{v}\cos A}{\sin A},\ WCV = \tfrac{1}{2}\pi - A,$$

Hence
$$x_3 = LB\cos BLM = Rw\cos(c + AM),$$
$$y_3 = LB\sin BLM = Rw\sin(c + AM),$$
$$z_3 = OL = R\dot{w},$$
$$x_2 = HW = GC\cos CGP = Rv\cos(b + AP),$$
$$y_2 = VW + HO = CW\cos A + HO$$
$$= R\{v\sin(b + AP)\cos A + \dot{v}\sin A\},$$
$$z_2 = CV = CW\sin A = R\{v\sin(b + AP)\sin A - \dot{v}\cos A\}.$$

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Also $\sin\frac{1}{2}BOC=\cos\alpha\sin\frac{1}{2}a$, or $\cos BOC=\cos^2\alpha\cos a+\sin^2\alpha$,

and $$R^2\cos BOC=x_1x_2+y_1y_2+z_1z_2.$$

Hence equating the two values for $\cos BOC$, and substituting for AM and AP from (17) and (18), we obtain finally, with the notation of (9),

$$u^2p+\dot{u}^2=(v^2q+\dot{v}^2)(w^2r+\dot{w}^2)$$
$$+\{\dot{q}\dot{r}-\dot{v}\dot{w}(1-q)(1-r)\}(\cos A+\dot{v}\dot{w}),$$
$$+\{\dot{q}\dot{w}(1-r)+\dot{r}\dot{v}(1-q)\}\sqrt{\{v^2w^2-(\cos A+\dot{v}\dot{w})^2\}}\ldots(19),$$

that is,

$$\cos^2\alpha\cos a+\sin^2\alpha=(\cos^2\beta\cos b+\sin^2\beta)(\cos^2\gamma\cos c+\sin^2\gamma)$$
$$+\{\sin b\sin c-\sin\beta\sin\gamma(1-\cos b)(1-\cos c)\}$$
$$\times(\cos A+\sin\beta\sin\gamma)$$
$$+\{\sin b\sin\gamma(1-\cos c)+\sin c\sin\beta(1-\cos b)\}$$
$$\times\sqrt{\{\cos^2\beta\cos^2\gamma-(\cos A+\sin\beta\sin\gamma)^2\}}\ldots(20).$$

5. Equation (20) may also be written

$$\cos a'=\cos b'\cos c'+\sin b'\sin c'\cos\left\{\cos^{-1}\left(\frac{\cos A+\sin\beta\sin\gamma}{\cos\beta\cos\gamma}\right)\right.$$
$$\left.-\cos^{-1}\left(\frac{\cos\beta\sin b}{\sin b'}\right)-\cos^{-1}\left(\frac{\cos\gamma\sin c}{\sin c'}\right)\right\}\ldots\ldots\ldots(21),$$

where $a'=BOC,\ldots,$ or a', b', c' are the sides BC, CA, AB of the great circles through A, B, C.

6. From (19) we may obtain, with the notation (9),

$$(\cos A+\dot{v}\dot{w})\{\dot{q}^2+\dot{v}^2(1-q)\}\{\dot{r}^2+\dot{w}^2(1-r)^2\}$$
$$=\{u^2p+\dot{u}^2-(v^2q+\dot{v}^2)(w^2r+\dot{w}^2)\}\{\dot{q}\dot{r}-\dot{v}\dot{w}(1-q)(1-r)\}$$
$$-\{\dot{q}\dot{w}(1-r)+\dot{r}\dot{v}(1-q)\}\sqrt{\{1-(u^2p+\dot{u}^2)^2-(v^2q+\dot{v}^2)^2}$$
$$-(w^2r+\dot{w}^2)^2+2(u^2p+\dot{u}^2)(v^2q+\dot{v}^2)(w^2r+\dot{w}^2)\}\ldots(22).$$

The sign of the root in (22) may be shewn to be negative, by writing down the equations corresponding to (19) for $\cos C$, and to (22) for $\cos B$, putting $C=\frac{1}{2}\pi$, $\beta=0$, $\gamma=0$, and comparing the value of $\cos B$ obtained from the two results with the value in (16). The root in (22) will be found to $=\dot{p}\dot{q}u$ in this case.

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7. In the case of a right-angled triangle, C being the right angle, we obtain from (19) for the relation between the sides

$$w^2r + \dot{w}^2 = \dot{v}^2p + \dot{u}^2q + \dot{p}\dot{q}\dot{u}\dot{v} + pq\,(u^2v^2 + \dot{u}^2\dot{v}^2)$$
$$+ \{\dot{p}\dot{v}\,(1-q) + \dot{q}\dot{u}\,(1-p)\}\,\sqrt{(u^2v^2 - \dot{u}^2\dot{v}^2)} \quad \ldots\ldots(23).$$

Also eliminating the root between (22) and the corresponding equation for $\cos C$, we have, after some reduction, with the notation of (9),

$$(\cos A + \dot{v}\dot{w})\,\{\dot{r}^2 + \dot{w}^2\,(1-r)^2\}\,\{\dot{p}\dot{v}\,(1-q) + \dot{q}\dot{u}\,(1-p)\}$$
$$= (u^2p + \dot{u}^2)\,\{\dot{p}\dot{q}\dot{r}\dot{v} - (1-p)\,\dot{q}\,(1-r)\,\dot{u}\dot{v}\dot{w} + \dot{p}q\,(1-r)\,\dot{w}$$
$$+ (1-p)\,(\dot{v}^2q + v^2)\,\dot{r}\dot{u}\}$$
$$- (w^2r + \dot{w}^2)\,\{\dot{p}\dot{q}\dot{r}v - (1-p)\,\dot{q}\,(1-r)\,\dot{u}\dot{v}\dot{w}$$
$$+ \dot{p}\,(\dot{v}^2q + v^2)\,(1-r)\,\dot{w} + (1-p)\,q\dot{r}\dot{u}\}$$
$$+ \dot{u}\dot{v}\,\{\dot{p}^2 + \dot{u}^2\,(1-p)^2\}\,\{\dot{q}\dot{w}\,(1-r) + \dot{r}v\,(1-q)\} \quad \ldots\ldots(24),$$

and a similar expression for $\cos B$.

8. Case $C = \frac{1}{2}\pi$, $\alpha = 0$, $\beta = 0$: or a right-angled triangle with the hypotenuse a little circle of latitude γ (fig. 50).

We cannot in this case use (24), but from (23) and (22) we obtain

$$w^2r + \dot{w}^2 = pq,$$

$$\cos A = \frac{\dot{p}\dot{q}\dot{r} - \dot{p}\dot{w}\,(1-r)}{\dot{r}^2 + \dot{w}^2\,(1-r)^2},$$

that is, $$\cos^2\gamma\cos c + \sin^2\gamma = \cos a\cos b \quad \ldots\ldots(25),$$

and $$\cos A = \frac{\cos a\sin b\sin c - \sin a\sin\gamma\,(1-\cos c)}{\sin^2 c + \sin^2\gamma\,(1-\cos c)^2} \quad \ldots(26),$$

which results may also be obtained independently.

9. Case $C = \frac{1}{2}\pi$, $\gamma = 0$: or a right-angled triangle with the sides containing the right angle little circles (fig. 51).

(24) gives in this case

$$\cos A = \frac{\dot{u}\,(1-p)\,\{u^2p + \dot{u}^2 + (1-q)\,\dot{v}^2 - qr\} + \dot{p}\dot{q}\dot{v}\,(u^2p + \dot{u}^2 - r)}{\dot{r}\,\{\dot{p}\dot{v}\,(1-q) + \dot{q}\dot{u}\,(1-p)\}}$$
$$\ldots\ldots(27).$$

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10. Case $C = \frac{1}{2}\pi$, $\beta = 0$: or a right-angled triangle with one side and the hypotenuse little circles (fig. 52).

In this case, (23) and (24) give

$$\cos^2\gamma\cos c + \sin^2\gamma = \cos a\cos(b+\alpha)\cos\alpha + \sin(b+\alpha)\sin\alpha \ldots (28),$$

$$\cos A =$$

$$\frac{\sin c\{\cos a\sin(b+\alpha)\cos\alpha - \cos(b+\alpha)\sin\alpha\} - \sin a(1-\cos c)\cos\alpha\sin\gamma}{\sin^2 c + \sin^2\gamma(1-\cos c)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

$$\cos B = \frac{\sin a\cos(b+\alpha)\sin c\cos\alpha - \sin\gamma(1-\cos c)\{\sin(b+\alpha)+\sin\alpha\}}{\sin^2 c + \sin^2\gamma(1-\cos c)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

11. Let ρ = the radius of the circle inscribed in a little-circle triangle ABC. Then with the figure of section 1, if I be the centre of the circle, IN the perpendicular on DE, and $DN = x$, we have

$$\sin x = \frac{\sin(\rho+\beta) + \sin(\rho+\gamma)\cos A}{\cos(\rho+\gamma)\sin A},$$

$$\sin(c + a_1 + b_2 - x) = \frac{\sin(\rho+\alpha) + \sin(\rho+\gamma)\cos B}{\cos(\rho+\gamma)\sin B}.$$

Eliminating x we have, putting $\cos(c + a_1 + b_2) = e$, $\sin(c + a_1 + b_2) = \dot{e}$,

$$\tan\rho(D^2 + 2eDF + F^2 - \dot{e}^2\dot{w}^2) =$$

$$-DE - FG - eDG - eEF - \dot{e}^2 w\dot{w} + \dot{e}\sqrt{\{(wD + \dot{w}E)^2}$$

$$+ 2e(wD + \dot{w}E)(wF + \dot{w}G) + (wF + \dot{w}G)^2 - (DG - EF)^2\},$$

where

$$D = \frac{v + w\cos A}{\sin A}, \quad E = \frac{\dot{v} + \dot{w}\cos A}{\sin A}, \quad F = \frac{u + w\cos B}{\sin B},$$

$$G = \frac{\dot{u} + \dot{w}\cos B}{\sin B}.$$

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Substituting for $\cos(c + a_1 + b_2)$ from (2), this becomes

$$\tan\rho\{2 - \Sigma\sin^2\alpha\sin^2 A + 2\Sigma\cos\beta\cos\gamma(\cos A + \cos B\cos C) + 2\cos A\cos B\cos C\}$$
$$= -\Sigma\sin\alpha\cos\alpha\sin^2 A - \Sigma\sin(\beta-\gamma)(\cos A + \cos B\cos C) + N\sqrt{[8\{\cos(\beta-\gamma)+\cos A\}\{\cos(\gamma-\alpha)+\cos B\}\{\cos(\alpha-\beta)+\cos C\}]} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

where N has its ordinary meaning (5).

12. Let P = the radius of the circle circumscribing ABC. Let K be the centre of the circle, and let the perpendiculars KL, KM, KN be drawn to BC, CA, AB. Then, if $BKL = \phi$, $CKM = \chi$, $AKN = \psi$, we have from (14)

$$\sin\phi = \frac{\sin\frac{1}{2}a\cos\alpha}{\sin P}.$$

Let $\sin^2 P = x$, $\sin\frac{1}{2}a\cos\alpha = f$, $\sin\frac{1}{2}b\cos\beta = g$, $\sin\frac{1}{2}c\cos\gamma = h$.

Then since $\phi + \chi + \psi = \pi$,

$$\cot\chi\cot\psi + \cot\psi\cot\phi + \cot\phi\cot\chi = 1.$$

Hence, if $\frac{x}{f^2} - 1 = U$, $\frac{x}{g^2} - 1 = V$, $\frac{x}{h^2} - 1 = W$,

$$\sqrt{(VW)} + \sqrt{(WU)} + \sqrt{(UV)} = 1,$$
$$VW + WU + UV - 1 = -2\sqrt{(UVW)}(\sqrt{U} + \sqrt{V} + \sqrt{W}),$$
$$(VW + WU + UV - 1)^2 = 4UVW(U + V + W + 2),$$
$$\left(x^2\Sigma\frac{1}{g^2h^2} - 2x\Sigma\frac{1}{f^2} + 2\right)^2 = 4\left(\frac{x^3}{f^2g^2h^2} - x^2\Sigma\frac{1}{g^2h^2} + x\Sigma\frac{1}{f^2} - 1\right)\left(x\Sigma\frac{1}{f^2} - 1\right),$$
$$x\left\{\frac{4}{f^2g^2h^2}\Sigma\frac{1}{f^2} - \left(\Sigma\frac{1}{g^2h^2}\right)^2\right\} = \frac{4}{f^2g^2h^2}.$$

Hence

$$\sin P = \frac{\sin\frac{1}{2}a\cos\alpha\,.\,\sin\frac{1}{2}b\cos\beta\,.\,\sin\frac{1}{2}c\cos\gamma}{2\sqrt{\{\sigma(\sigma - \sin\frac{1}{2}a\cos\alpha)(\sigma - \sin\frac{1}{2}b\cos\beta)(\sigma - \sin\frac{1}{2}c\cos\gamma)\}}} \quad \ldots\ldots\ldots\ldots\ldots (32),$$

where $2\sigma = \sin\frac{1}{2}a\cos\alpha + \sin\frac{1}{2}b\cos\beta + \sin\frac{1}{2}c\cos\gamma$.

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13. If $ABCD$ be a great-circle quadrilateral, having B and D right angles, and $BC=b$, $CD=c$, its area is (fig. 53)

$$R^2\left\{A-\cos^{-1}\left(\frac{\cos A+\sin b\sin c}{\cos b\cos c}\right)\right\} \quad \ldots\ldots\ldots(33).$$

For its area $=R^2(A+C-\pi)$. Let $AB=a$, $DA=d$; then

$$\cos a\cos b=\cos c\cos d,$$

$$\cos b\cos c+\sin b\sin c\cos C=\cos a\cos d+\sin a\sin d\cos A$$

$$=\frac{\cos^2 a\cos b}{\cos c}+\sin a\sin d\cos A;$$

and, as in (1),

$$\sin a=\frac{\sin c+\sin b\cos A}{\cos b\sin A}, \quad \sin d=\frac{\sin b+\sin c\cos A}{\cos c\sin A},$$

whence $$\cos C=-\frac{\cos A+\sin b\sin c}{\cos b\cos c}.$$

14. The area of a spherical belt bounded by a great circle, and a little circle of latitude α is well known to $=2\pi R.R\sin\alpha$. Hence in section 1, the area of BCC_2B_1 is $R^2.a\sin\alpha$.

Hence the area of the little-circle triangle ABC is

$$R^2\left\{\Sigma\cos^{-1}\left(\frac{\cos A+\sin\beta\sin\gamma}{\cos\beta\cos\gamma}\right)-\Sigma a\sin\alpha-\pi\right\} \ldots(34).$$

END OF VOL. XXVII.

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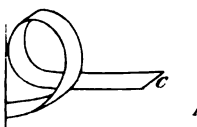


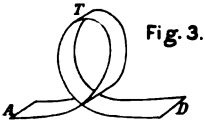
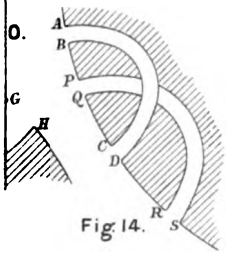


Fig. 3.

Fig. 5.



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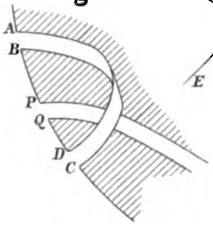

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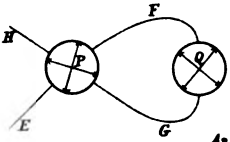

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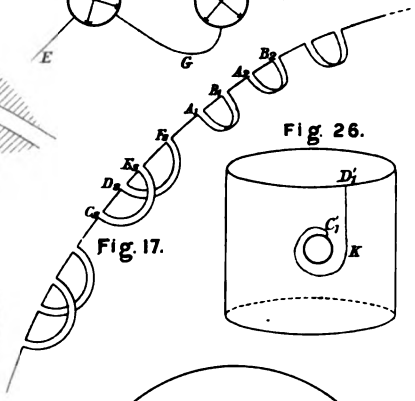
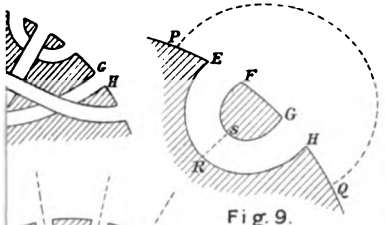


Fig. 17.

Fig. 26.



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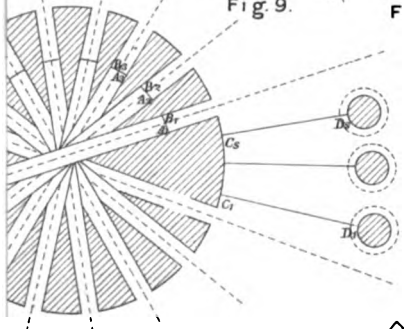

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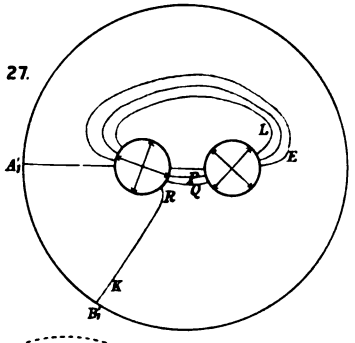

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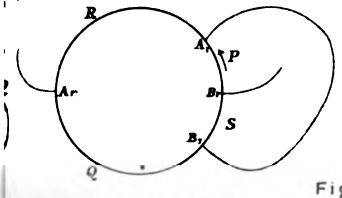


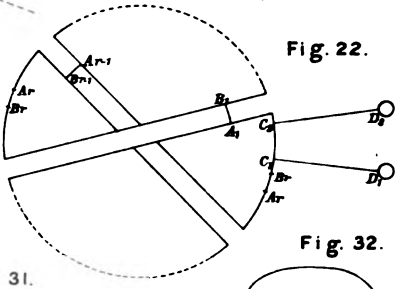
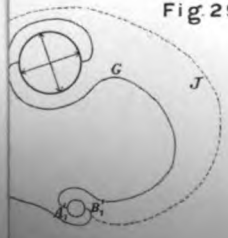


Fig. 22.

Fig. 29.



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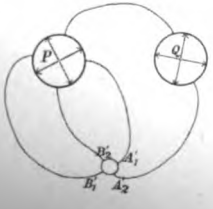

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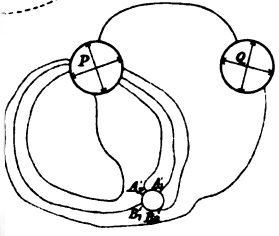

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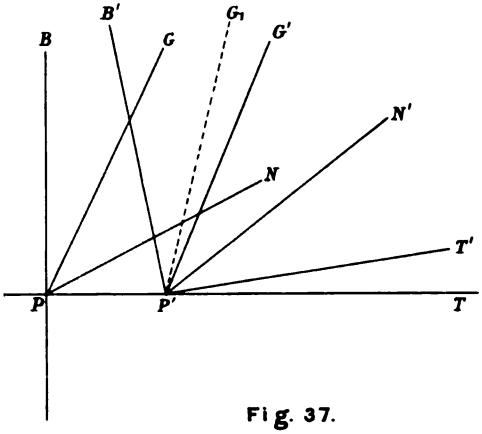

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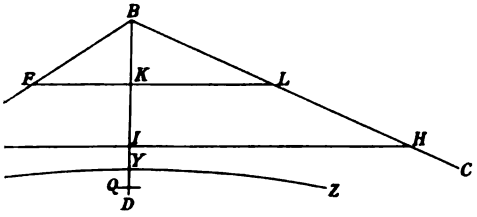

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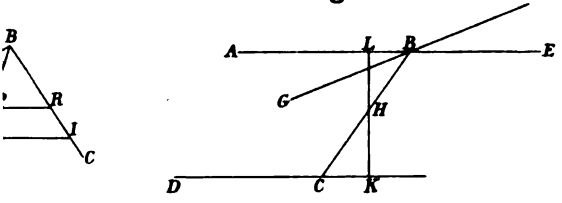

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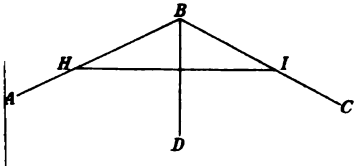

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